Infinitely many conservation laws for the discrete KdV equation

Alexander Rasin and Jeremy Schiff

Mathematics Department
Bar-Ilan University, Israel

March 18, 2009
Outline

Conservation laws

Methods of generation of conservation laws.

Infinitely many conservation laws for dKdV.
Conservation laws for a scalar PDE

A conservation law is an expression of the form

\[ \partial_t G + \partial_x F = 0 \]

For the KdV equation

\[ u_t = \frac{1}{4} u_{xxx} + 3 uu_x \]

we have

\[ \partial_t (u) + \partial_x \left( -\frac{1}{4} u_{xx} - \frac{3}{2} u^2 \right) = 0 , \]

\[ \partial_t (u^2) + \partial_x \left( -\frac{1}{2} uu_{xx} + \frac{1}{4} u_x^2 - 2u^3 \right) = 0 , \]

\[ \partial_t \left( 4u^3 - u_x^2 \right) + \partial_x \left( -9u^4 + \frac{1}{2} u_x u_{xxx} - \frac{1}{4} u_{xx}^2 - 3u^2 u_{xx} + 6uu_x^2 \right) = 0 . \]

Trivial conservation law

\[ F = F_0 - \partial_t f \]

\[ G = G_0 + \partial_x f \]

where \( F_0, G_0 \) both vanish as a consequence of the equation.
Equations on quad–graph

A quad-graph equation

\[ P(k, l, u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}, a) = 0 \]

A conservation law is an expression of the form

\[ (S_l - l)G + (S_k - l)F = 0 \]

Trivial conservation law

\[ F = F_0 - (S_l - l)f \]
\[ G = G_0 + (S_k - l)f \]

where \( F_0, G_0 \) both vanish as a consequence of the equation
How to check if the conservation law is trivial?

Trivial conservation law

\[ F = F_0 - (S_l - I)f \]
\[ G = G_0 + (S_k - I)f \]

**Figure:** Possible points of $G$
Conservation law on the horizontal (vertical) line

Definition
We say the conservation law for the quad-graph equation is “on the horizontal line” if \( G \) depends only on values of \( u_{n,m} \) with \( m = 0 \), and “on the vertical line” if \( F \) depends only on values of \( u_{n,m} \) with \( n = 0 \).

Lemma
For the dKdV equation, a trivial conservation law on the horizontal (vertical) line can always be presented in a form with \( G = (S_k - I)f \) (\( F = (S_l - I)f \)), where \( f \) is a function on the horizontal (vertical) line.
Gardner method for the continuous KdV

The Bäcklund transformation

\[ v_x = \theta - 2u - v^2 \]
\[ v_t = -\frac{1}{2}u_{xx} + (\theta + u) v_x + u_x v \]

Initial conservation law:

\[ \partial_t v + \partial_x \left( \frac{1}{2}u_x - (u + \theta)v \right) = 0. \]

Expanding in a suitable series in \( \theta \) yields

\[ v = \theta^{1/2} - \frac{u}{\theta^{1/2}} + \frac{u_x}{2\theta} - \frac{u_{xx} + 2u^2}{4\theta^{3/2}} + \frac{u_{xxx} + 8uu_x}{8\theta^2} - \frac{u_{xxxx} + 8u^3 + 10u_x^2 + 12uu_{xx}}{16\theta^{5/2}} + O\left(\theta^{-3}\right). \]
Gardner method for the discrete KdV

The discrete KdV is

$$(u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + \beta - \alpha = 0.$$ 

The Bäcklund transformation for dKdV

$$(\tilde{u}_{0,0} - u_{0,1})(u_{0,0} - \tilde{u}_{0,1}) = \theta - \beta,$$

$$(\tilde{u}_{0,0} - u_{1,0})(u_{0,0} - \tilde{u}_{1,0}) = \theta - \alpha.$$ 

In the case $\theta = \beta$ we can take $\tilde{u}_{0,0} = u_{0,1}$ or $u_{0,-1}$, and in the case $\theta = \alpha$ we can take $\tilde{u}_{0,0} = u_{1,0}$ or $u_{-1,0}$. Consider the case $\theta = \alpha + \epsilon$ and

$$\tilde{u}_{0,0} = u_{1,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i.$$
Gardner method for the discrete KdV

The leading order approximation gives

\[ v_{0,0}^{(1)} = \frac{1}{u_{0,0} - u_{2,0}} . \]

Higher order terms give

\[ v_{0,0}^{(i)} = \frac{1}{u_{0,0} - u_{2,0}} \sum_{j=1}^{i-1} v_{0,0}^{(j)} v_{1,0}^{(i-j)} , \quad i = 2, 3, \ldots . \]

Initial conservation law

\[ F = - \ln \left( \tilde{u}_{0,0} - u_{0,1} \right) , \quad G = \ln \left( \tilde{u}_{0,0} - u_{1,0} \right) \]
Gardner method for the discrete KdV

By expanding $F$ and $G$ in powers of $\epsilon$ we obtain

$$F = -\ln \left( u_{1,0} - u_{0,1} + \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i \right) = -\ln (u_{1,0} - u_{0,1})$$

$$- \ln \left( 1 + \frac{1}{u_{1,0} - u_{0,1}} \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i \right),$$

$$G = \ln \left( \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i \right) = \ln \epsilon - \ln (u_{0,0} - u_{2,0}) + \ln \left( 1 + \frac{1}{v_{0,0}^{(1)}} \sum_{i=1}^{\infty} v_{0,0}^{(i+1)} \epsilon^i \right).$$
New conservation laws

Notation

\[ A_i = S_k^i \left( \frac{1}{u_{0,0} - u_{2,0}} \right) \, , \quad i = 0, 1, 2, \ldots \, , \quad B = \frac{1}{u_{1,0} - u_{0,1}} \, , \]

we obtain

\[
\begin{align*}
F_0 &= \ln B \\
G_0 &= \ln A_0 \\
F_1 &= -BA_0 \\
G_1 &= A_0A_1 \\
F_2 &= -A_0^2A_1B + \frac{1}{2}A_0^2B^2 \\
G_2 &= A_0A_1^2A_2 + \frac{1}{2}A_0^2A_1^2 \\
F_3 &= -A_0^2A_1^2A_2B - A_0^3A_1^2B + A_0^3A_1B^2 - \frac{1}{3}A_0^3B^3 \\
G_3 &= A_0A_1^2A_2^2A_3 + A_0^2A_1^3A_2 + A_0A_1^3A_2^2 + \frac{1}{3}A_0^3A_1^3
\end{align*}
\]
Gardner method for other quad-graphs

BT for modified dKdV equation:

\[ \alpha(u_{0,0}u_{1,0} + \tilde{u}_{0,0}\tilde{u}_{1,0}) - \theta(u_{0,0}\tilde{u}_{0,0} + u_{1,0}\tilde{u}_{1,0}) = 0, \]
\[ \theta(u_{0,0}\tilde{u}_{0,0} + u_{0,1}\tilde{u}_{0,1}) - \beta(u_{0,0}u_{0,1} + \tilde{u}_{0,0}\tilde{u}_{0,1}) = 0 \]

Initial conservation law

\[ f = \ln \left( \frac{\theta u_{0,1} - \beta \tilde{u}_{0,0}}{u_{0,0}} \right), \quad g = -\ln \left( \frac{-\theta u_{1,0} + \alpha \tilde{u}_{0,0}}{u_{0,0}} \right). \]

BT for cross-ratio equation:

\[ \alpha(u_{0,0} - \tilde{u}_{0,0})(u_{1,0} - \tilde{u}_{1,0}) - \theta(u_{0,0} - u_{1,0})(\tilde{u}_{0,0} - \tilde{u}_{1,0}) = 0, \]
\[ \theta(u_{0,0} - u_{0,1})(\tilde{u}_{0,0} - \tilde{u}_{0,1}) - \beta(u_{0,0} - \tilde{u}_{0,0})(u_{0,1} - \tilde{u}_{0,1}) = 0. \]

Initial conservation law

\[ f = \ln \left( \frac{\theta(u_{0,1} - u_{0,0}) + \beta(u_{0,0} - \tilde{u}_{0,0})}{u_{0,1} - u_{0,0}} \right), \]
\[ g = -\ln \left( \frac{\theta(u_{1,0} - u_{0,0}) + \alpha(u_{0,0} - \tilde{u}_{0,0})}{u_{1,0} - u_{0,0}} \right). \]
**Symmetry method**

An infinitesimal symmetry for the quad-graph is

\[
\begin{align*}
    u_{0,0} & \rightarrow \hat{u}_{0,0} = u_{0,0} + \epsilon Q(k, l, u, a) + O(\epsilon^2) , \\
    a & \rightarrow \hat{a} = a + \epsilon \xi(a) + O(\epsilon^2) ,
\end{align*}
\]

Therefore

\[
\hat{u}_{i,j} = u_{i,j} + \epsilon S^i_k S^j_l Q + O(\epsilon^2),
\]

Infinitesimal generator

\[
X = Q \frac{\partial}{\partial u_{0,0}} + \xi \cdot \frac{\partial}{\partial a}.
\]

Prolonged infinitesimal generator

\[
\hat{X} = \sum_{i,j} S^i_k S^j_l(Q) \frac{\partial}{\partial u_{i,j}} + \xi \cdot \frac{\partial}{\partial a}.
\]

Linearized symmetry conditions

\[
\hat{X}(P) = 0
\]
Symmetry method

Prolonged infinitesimal generator commutes with shift operators

\[ [\hat{X}, S_k] = 0, \quad [\hat{X}, S_l] = 0. \]

So if \( F \) and \( G \) are components of a conservation law

\[ (S_k - I)F + (S_l - I)G = 0 \mid_{P=0}, \]

then by applying the symmetry generator we obtain

\[ 0 = \hat{X}((S_k - I)F) + \hat{X}((S_l - I)G) \mid_{P=0} = (S_k - I)\hat{X}(F) + (S_l - I)\hat{X}(G) \mid_{P=0}. \]

Thus

\[ F_{\text{new}} = \hat{X}(F), \quad G_{\text{new}} = \hat{X}(G), \]

is also a conservation law.
Symmetry method

Known symmetry generators for dKdV include

\[ X_0 = \frac{1}{u_{1,0} - u_{-1,0}} \frac{\partial}{\partial u_{0,0}}, \quad Y_0 = \frac{1}{u_{0,1} - u_{0,-1}} \frac{\partial}{\partial u_{0,0}}, \]
\[ X = \frac{k}{u_{1,0} - u_{-1,0}} \frac{\partial}{\partial u_{0,0}} - \partial \alpha, \quad Y = \frac{l}{u_{0,1} - u_{0,-1}} \frac{\partial}{\partial u_{0,0}} - \partial \beta. \]

Known conservation laws include

\[ F = \ln(u_{0,1} - u_{-1,0}), \quad G = -\ln(u_{1,0} - u_{-1,0}), \]
\[ \bar{F} = \ln(u_{0,1} - u_{0,-1}), \quad \bar{G} = -\ln(u_{1,0} - u_{0,-1}), \]
\[ \tilde{F} = kF + l\bar{F}, \quad \tilde{G} = kG + l\bar{G}. \]
Symmetry method

Theorem
The dKdV equation has an infinite number of nontrivial conservation laws on the horizontal line, generated by repeated application of the symmetry $X$ to the conservation law with components $(F, G)$, and an infinite number on the vertical line, generated by repeated application of the symmetry $Y$ to the conservation law with components $(\bar{F}, \bar{G})$.

We show that

$$E(G_n) = (S_k - S_k^{-1})Q_n, \quad n = 0, 1, 2, \ldots,$$

where $Q_n$ is the characteristic of $X_n = [X, X_{n-1}]$ and $E$ is the discrete Euler operator

$$E(A) = \sum_{n,m} S_k^{-n}S_l^{-m} \left( \frac{\partial A}{\partial u_{n,m}} \right).$$
Symmetry method

For mKdV equation we have

\[ E(G_n) = \frac{1}{u_{1,0}u_{0,0}} S_k Q_n - \frac{1}{u_{-1,0}u_{0,0}} S_k^{-1} Q_n, \quad n = 0, 1, 2, \ldots , \]

where \( Q_n \) is the characteristic of \( X_n = [X, X_{n-1}] \).

For cross-ratio equation we have

\[ E(G_n) = \frac{S_k Q_n}{(u_{1,0} - u_{0,0})^2} - \frac{S_k^{-1} Q_n}{(u_{0,0} - u_{-1,0})^2}, \quad n = 0, 1, 2, \ldots , \]

where \( Q_n \) is the characteristic of \( X_n = [X, X_{n-1}] - X_{n-1} \).
The continuum limit

By replacing

\[ u_{i,j} = u(x + ih, t + jh), \quad \alpha = \alpha(h), \quad \beta = \beta(h), \]

and taking the limit we obtain

\[ u_x^2 - u_t^2 = C. \quad (1) \]

The continuum limits of the symmetries

\[ X_0 = \frac{1}{2u_x} \partial_u, \quad Y_0 = \frac{1}{2u_t} \partial_u, \]
\[ X = \frac{x}{2u_x} \partial_u + \partial_C, \quad Y = \frac{t}{2u_t} \partial_u + \partial_C. \]

The continuum limits of the conservation laws

\[ F = \ln(u_x + u_t), \quad G = -\ln(u_x), \]
\[ \tilde{F} = \ln(u_t), \quad \tilde{G} = -\ln(u_x + u_t), \]
\[ \tilde{F} = xF + t\tilde{F}, \quad \tilde{G} = xG + t\tilde{G}. \]
Theorem
The equation (1) has an infinite number of distinct, nontrivial conservation laws generated by repeated application of the symmetry $X$ to the conservation law with components $(F, G)$. Writing $F_n = X^n(F)$, $G_n = X^n(G)$ we find $G_n = \frac{1}{u_x^{2n}}$ for $n \geq 1$ (up to addition of a trivial conservation law and rescaling).

Theorem
The continuum limit of the conservation laws constructed using the Gardner transformation coincides with those constructed by the symmetry method.
The continuum limit

The continuum limit of mKdV is

\[ u_x^2 - u_t^2 - Cu^2 = 0 \]

This equation can be obtained from (1) by substitution

\[ u = \ln(v) \]

The continuum limit of cross-ratio equation

\[ u_x^2 - Cu_t^2 = 0 \]

The continuum limits of the conservation laws

\[ F = \ln(u_x) - \ln(u_x + u_t), \quad G = 0, \]
\[ \bar{F} = 0, \quad \bar{G} = \ln(u_t) - \ln(u_x + u_t). \]
Open problems

- Construction of Bäcklund transformations and initial conservation laws;
- Connection between Gardner and symmetry methods;
- Realization of Gelfand-Dikii method;
- Importance of conservation laws, bounds of norms of solutions.

**Rasin, A. G., and Schiff, J.**
Infinitely many conservation laws for the discrete KdV equation.
arXiv:0901.0390v1 [nlin.SI].