Quiver Presentations of Descent Algebras of finite Coxeter Groups.

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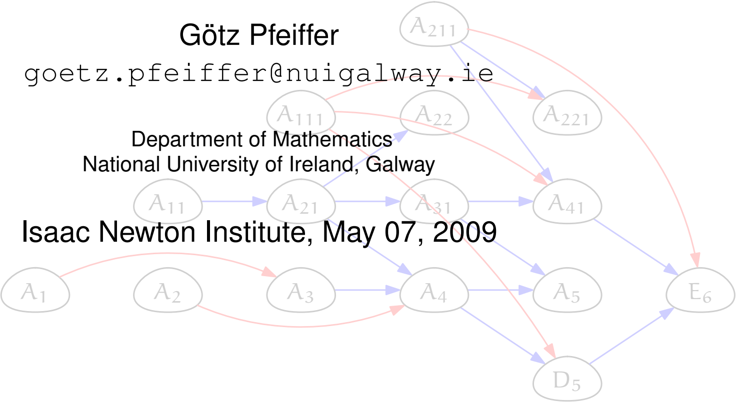


Finite Coxeter Groups.

- Let W be a **Coxeter group**, that is a group with
  - **generators** $S = \{s, t, \dots\}$ and
  - **relations** $(st)^{m(s,t)} = 1$, for certain $m(s,t) = m(t,s) \in \{2, 3, \dots, \infty\}$, if $s \neq t$, and $m(s,s) = 1$.

Examples

- $\mathrm{Sym}(n) = \langle (i, i+1) : i = 1, \dots, n-1 \rangle$.
- $D_{2m} = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle$.

- **Length function**: $\ell(ws) = \ell(w) \pm 1$, for $w \in W$, $s \in S$.
- Assume W is **finite**. Then W has a **longest element** w_0.

Distinguished Coset Representatives.

- For $J \subseteq S$, the **standard parabolic subgroup** $W_J = \langle J \rangle$ of W is a Coxeter group.
- $X_J := \{w \in W : \ell(sw) > \ell(w) \text{ for all } s \in J\}$ is a **right transversal** of W_J in $W = W_J \cdot X_J$ consisting of the unique **elements of minimal length** in each coset.
- $X_J^{-1} := \{x^{-1} : x \in X_J\}$ is a minimal length **left transversal** of W_J in W.
- $X_J^K := X_J \cap W_K$, for $J \subseteq K \subseteq S$, is a minimal length right transversal of W_J in W_K.
- $X_{JK} = X_J \cap X_K^{-1}$, for $J, K \subseteq S$, is a set of minimal length **double coset representatives** of W_J and W_K in $W = \bigsqcup_{d \in X_{JK}} W_J d W_K$.
- If $d \in X_{JK}$ then $W_J^d \cap W_K = W_L$, where $L = J^d \cap K$.

Solomon's Theorem.

- $X_{JKL} := \{d \in X_{JK} : J^d \cap K = L\}$.
- $x_J := \sum X_J^{-1} \in \mathbb{Q}[W]$.
- Then $x_J x_K = \sum_{L \subseteq S} a_{JKL} x_L$, where $a_{JKL} = |X_{JKL}| \in \mathbb{Z}$.
- The subspace $\Sigma(W) := \langle x_J : J \subseteq S \rangle_{\mathbb{Q}}$ is subalgebra of $\mathbb{Q}[W]$, called the **descent algebra** of W.
- The map $\theta \colon x_J \mapsto 1_{W_J}^W$, assigning to x_J the **permutation character** of the action of W on the cosets of W_J, is an **algebra homomorphism**.
- θ has a **commutative image** $\operatorname{img} \theta = \mathbb{Q}^l \subseteq \mathbb{Q}^r$, the character ring of W.
- θ has a **nilpotent kernel** $\ker \theta = \langle x_J - x_K : J \sim K \rangle$ coinciding with the radical of $\Sigma(W)$.
- Hence $\Sigma(W)$ is a **basic algebra**. As such it has a **presentation** as a **quiver with relations**. **Find it!**

Descent Classes.

- $\mathcal{D}(w) := \{s \in S : \ell(sw) < \ell(w)\}$
  is the **descent set** of $w \in W$.
- W is **partitioned** into 2^n **descent classes**
  $Y_K := \{w \in W : \mathcal{D}(w) = S \setminus K\} = \mathcal{D}^{-1}(S \setminus K)$, $K \subseteq S$,
  where $n = |S|$.
- For example, $Y_S = \{\text{id}\}$ and $Y_\varnothing = \{w_0\}$.
- Then $X_J = \bigsqcup_{K \supseteq J} Y_K = \{w \in W : \mathcal{D}(w) \cap J = \varnothing\}$.
- The 2^n elements $y_K := \sum Y_K^{-1}$ form a basis of $\Sigma(W)$.
- $x_J = \sum_{K \supseteq J} y_K$, and $y_K = \sum_{J \supseteq K} (-1)^{|J-K|} x_J$.
- In particular,
  $w_0 = \sum_J (-1)^{|J|} x_J$ and $\epsilon_W = \theta(w_0) = \sum_J (-1)^{|J|} 1_{W_J}^W$.

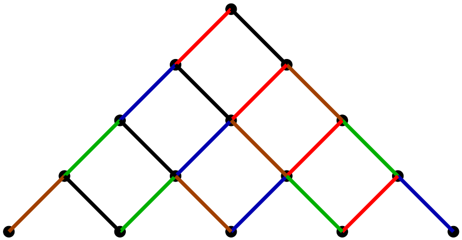
The Coset Graph.

Represent X_J as a **directed colored graph** with

- **vertices**: $x \in X_J$ and
- **edges**: $x \xrightarrow{s} xs$ if $\ell(xs) > \ell(x)$, $s \in S$.

Example ($\mathrm{Sym}(6) = \langle s_1, s_2, s_3, s_4, s_5 \rangle \geq \mathrm{Sym}(5)$)

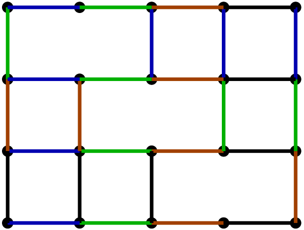
Example ($\mathrm{Sym}(6) \geq \mathrm{Sym}(2) \times \mathrm{Sym}(4)$)



Properties of Coset Graphs.

1. **Transitivity**: Let $J \subseteq K \subseteq S$. Then $X_J = X_K \cdot X_J^K$.
   ($W_J \cdot X_J = W = W_K \cdot X_K = W_J \cdot X_J^K \cdot X_K$.)

Example ($\mathrm{Sym}(5) \geq \mathrm{Sym}(3)$)

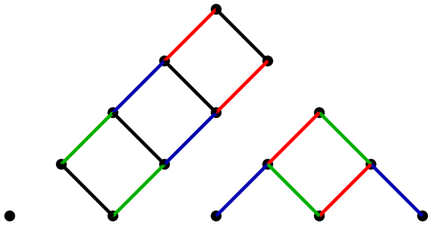


2. **Shift:** Let $J, K \subseteq S$. Then $dX_{J^d \cap K} = X_{J \cap {}^d K}$ for all $d \in X_{JK}$.

Mackey Decomposition.

3. **Mackey-Decomposition**: Let $J, K \subseteq S$. Then
   $X_J = \bigsqcup_{d \in X_{JK}} d \cdot X^J_{J^d \cap K}$.

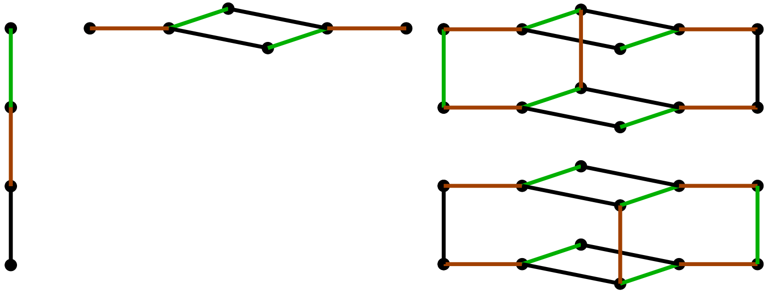
Example ($\mathrm{Sym}(6) \geq \mathrm{Sym}(2) \times \mathrm{Sym}(4)$)



Proof of Solomon's Theorem.

$$x_J\, x_K \overset{(3)}{=} x_J \sum_{d \in X_{JK}} x^J_{J \cap {}^d K} d \overset{(1)}{=} \sum_{d \in X_{JK}} x_{J \cap {}^d K} d \overset{(2)}{=} \sum_{d \in X_{JK}} x_{J^d \cap K}.$$

Example



A Basis of Idempotents.

Theorem (Bergeron, Bergeron, Howlett and Taylor 1992)

$\Sigma(W)$ *has a basis* $\{e_L : L \subseteq S\}$ *such that:*

- e_L *is* ***idempotent*** *for all* $L \subseteq S$;
- *the radical of* $\Sigma(W)$ *is generated by the* ***differences*** $e_J - e_K$, $J \sim K$;
- *the* ***class sums*** $\epsilon_\lambda = \sum_{L \in \lambda} e_L$ *over the* ***shapes*** $\lambda \in \Lambda$ *form a complete set of pairwise orthogonal* ***primitive idempotents*** *of* $\Sigma(W)$.

Still Wanted:

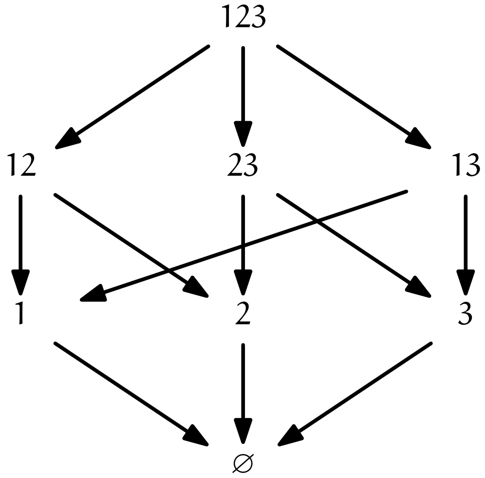
a presentation of $\Sigma(W)$ as a **quiver with relations**.

Quivers and Relations.

- A **quiver** is a directed multigraph $Q = (V, E)$.
- The **path algebra** A of Q is the $\mathbb{Q}$-space spanned by all oriented paths in Q, with **path concatenation** as multiplication.
- Denote by $\mathcal{A}_l$ the set of paths of length l, by A_l their span.
- $A = \bigoplus_{l \geq 0} A_l$ is **graded** by path length; $\mathcal{A}_0 = V$, $\mathcal{A}_1 = E$.
- The elements $v \in \mathcal{A}_0$ form a complete set of **primitive orthogonal idempotents** for A.
- An ideal $I \subseteq A$ is **admissible**, if $I \subseteq J^2$, where J is the ideal generated by $\mathcal{A}_1 = E$.
- A basic algebra has a presentation (Q, I) as path algebra of a unique quiver Q modulo some admissible ideal I.

For Example . . .

- Let Q be the **Hasse diagram** of $(\mathcal{P}(S), \supseteq)$.
- For example, $S = \{1, 2, 3\}$:



The Plan.

- Given (W, S).
- Let A be the path algebra of this quiver $Q = (\mathcal{P}(S), \supset)$.
- Present $\Sigma(W)$ as a quotient of a subalgebra of A.

More precisely:

- Define an action of the free monoid S^* on the paths $\mathcal{A}$.
- Partition $\mathcal{A}$ into S^*-orbits $\mathcal{X} = \mathcal{A}/S^*$
- Let Ξ be the subspace of A spanned by the orbit sums.
- Show that Ξ is a subalgebra of A.
- Associate to each $a \in \mathcal{A}_l$ an edge of the S^*-action graph on $\mathcal{A}_{l-1}$.
- Identify a with the difference $\delta(a)$ of the end points of that edge: $a \equiv \delta(a)$.
- Show that $\Xi/{\equiv}$ is (anti-)isomorphic to $\Sigma(W)$.

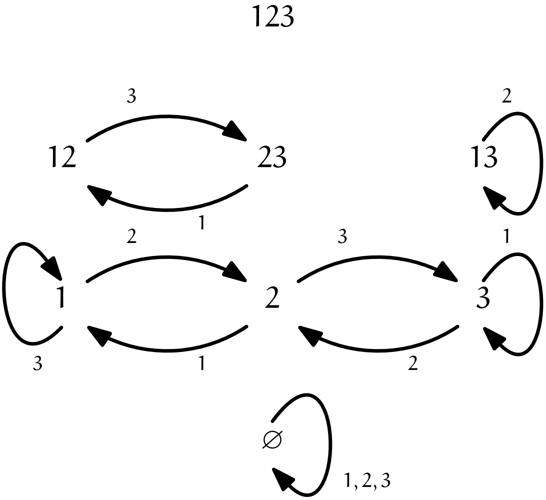
Action.

- The **conjugation action** of W on its subsets **partitions** the power set $\mathcal{P}(S)$ into **shapes** $[J] = \{K \subseteq S : K \sim J\}$.
- Conjugation by the **longest element** w_J of W_J permutes J.
- If $J \subseteq L \subseteq S$ and $d = w_J w_L$, the **longest coset representative** of W_J in W_L, then $J^d \subseteq L$.
- The **free monoid S^* acts** on $\mathcal{P}(S)$ via $J.r = \{s^d : s \in J\}$, for $J \subseteq S$, $r \in S$, where $L = J \cup \{r\}$ and $d = w_J w_L$.

Theorem
$[J] = J.S^*$.

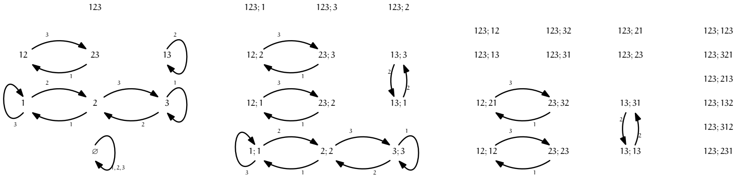
- Furthermore, this action can be used to describe the **normalizer** of W_J in W as a semidirect product $N_W(W_J) = W_J \rtimes N_J$, where $N_J = X_{JJJ}$ is the **fundamental group** of the S^*-action graph on $[J]$ with base point J.
- Denote by $\Lambda = \{[J] : J \subseteq S\} = \mathcal{P}(S)/S^*$ the shapes of W.

For Example A_3.



Alleys.

- Recall $Q = (\mathcal{P}(S), \supset\!\!\!\cdot\,)$.
- Denote $L_s := L \setminus \{s\}$.
- An **alley** is a path $L \to L_s \to L_{s,t} \to \cdots \to L_{s,t,\ldots}$ in Q, i.e., a **pair** $(L; s, t, \ldots) \in \mathcal{A}$, where $L \subseteq S$ and $s, t, \ldots \in L$ are pairwise different.
- The free monoid S^* acts on $\mathcal{A}$: $(L; s, t, \ldots).r = (L^d; s^d, t^d, \ldots)$, where $d = w_L w_M$, for $M = L \cup \{r\}$.
- The **vertices** of $\mathcal{A}_l$ are in **bijection** to the **edges** on $\mathcal{A}_{l-1}$: $(L; s, t, \ldots) \mapsto (L_s; t, \ldots) \xrightarrow{s} (L_s; t, \ldots).s$.
- For example, in type A_3:



Streets.

- The action of S^* partitions $\mathcal{A}$ into **streets**.
- Denote by $[\mathfrak{a}]$ the S^*-orbit of $\mathfrak{a} \in \mathcal{A}$ and let $\mathcal{X} = \mathcal{A}/S^*$.
- Identify a street with the sum of its elements in $A = \mathbb{Q}[\mathcal{A}]$.

Theorem

A product of streets is a sum of streets: If $\mathfrak{a} \in \mathcal{A}$ is an alley from L to L' and $\mathfrak{a}' \in \mathcal{A}$ is an alley from L' to L'' then

$$[\mathfrak{a}] \circ [\mathfrak{a}'] = \sum_{d \in D} [\mathfrak{a} \circ (\mathfrak{a}')^d],$$

where D is a set of double coset representatives of the stabilizer of $\mathfrak{a}$ and the stabilizer of $\mathfrak{a}'$ in the normalizer complement $N_{L'}$.

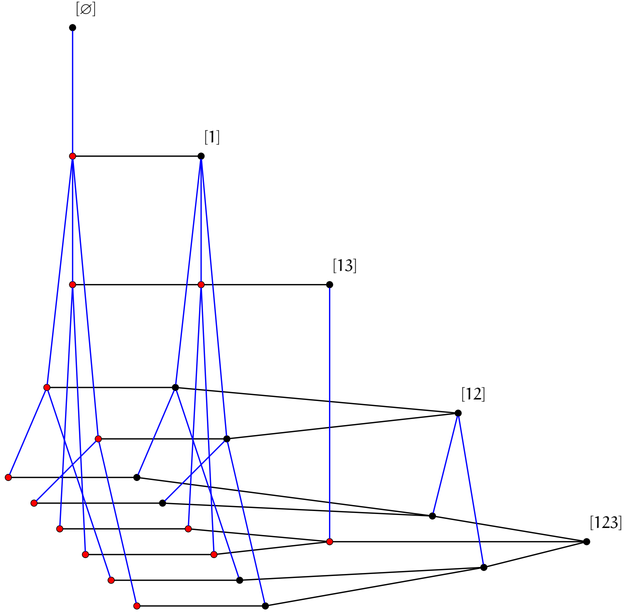
Alternatively, ...

- Let $\mathfrak{a}, \mathfrak{a}' \in \mathcal{A}$ be alleys.
- Write $\mathfrak{a}' \preceq_\pi \mathfrak{a}$ ("$\mathfrak{a}'$ is a **prefix** of $\mathfrak{a}$" or "$\mathfrak{a}$ begins with $\mathfrak{a}'$") if $\mathfrak{a} = \mathfrak{a}' \circ \mathfrak{a}''$ for some $\mathfrak{a}'' \in \mathcal{A}$.
- Write $\mathfrak{a} \succeq_\sigma \mathfrak{a}'$ ("$\mathfrak{a}'$ is a **suffix** of $\mathfrak{a}$" or "$\mathfrak{a}$ ends in $\mathfrak{a}'$") if $\mathfrak{a} = \mathfrak{a}'' \circ \mathfrak{a}'$ for some $\mathfrak{a}'' \in \mathcal{A}$.
- Then $\mathfrak{a} \circ \mathfrak{a}' = \sum_{\substack{L \subseteq S, \mathfrak{a}'' \in \mathcal{A} \\ \mathfrak{a} \succeq_\sigma L \preceq_\pi \mathfrak{a}' \\ \mathfrak{a} \preceq_\pi \mathfrak{a}'' \succeq_\sigma \mathfrak{a}'}} \mathfrak{a}''$.
- These partial orders are compatible with the action of S^*: write $[\mathfrak{a}'] \preceq_\pi [\mathfrak{a}]$ if $\mathfrak{a}' \preceq_\pi \mathfrak{a}$, ...

Theorem

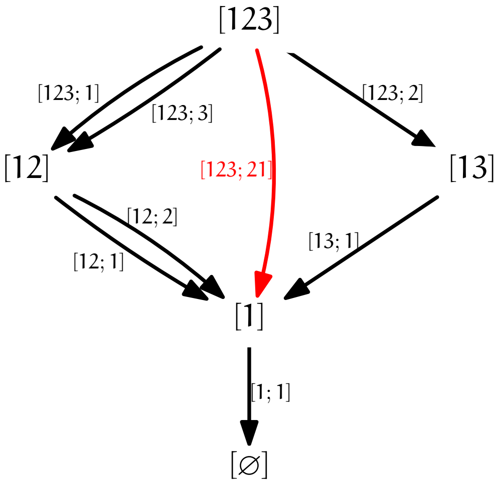
Let $\alpha, \alpha' \in \mathcal{X}$ be streets. Then $\alpha \circ \alpha' = \sum_{\substack{\lambda \in \Lambda, \alpha'' \in \mathcal{X} \\ \alpha \succeq_\sigma \lambda \preceq_\pi \alpha' \\ \alpha \preceq_\pi \alpha'' \succeq_\sigma \alpha'}} \alpha''$.

For Example A_3.



A conjecture.

- The streets algebra Ξ is a path algebra.
- For example, in type A_3:



Differences.

- Denote by $\delta(L; s, t, \ldots) = (L_s; t, \ldots) - (L_s; t, \ldots).s$ the **difference** of the endpoints of directed edge $(L_s; t, \ldots) \longrightarrow (L_s; t, \ldots).s$ in the graph of the S^*-action.
- Impose on $\mathcal{A}$ the **relations** $(L; s, t, \ldots) \equiv \delta(L; s, t, \ldots)$.
- Denote by $\overline{(L; s, t, \ldots)}$ the image of $(L; s, t, \ldots)$ in $\mathcal{A}/\equiv$.

Theorem

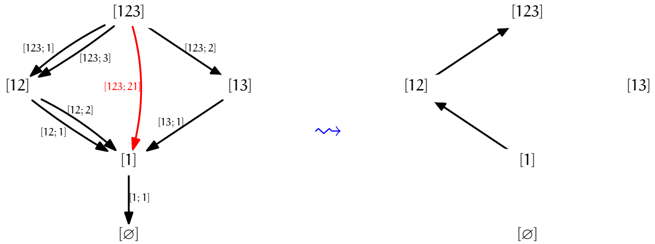
*$\mathcal{A}/\equiv \; = \Xi/\equiv$ with basis $\{\overline{(L; \varnothing)}; L \subseteq S\}$. Moreover, the shapes $\lambda = \sum_{L \in \lambda} \overline{(L; \varnothing)} = \overline{[L; \varnothing]}$, $\lambda \in \Lambda$, form a complete set of pairwise orthogonal **primitive idempotents** of $\Xi/\equiv$;*

The Main Theorem.

Theorem (Pf)

The restriction to Ξ of the linear map $L \mapsto e_L$ for $L \subseteq S$ is a surjective algebra homomorphism from Ξ to $\Sigma(W)^{op}$, inducing a bijection $\lambda \mapsto \epsilon_\lambda$, $\lambda \in \Lambda$, on sets of primitive idempotents.

- Hence every subset $\mathcal{E} \subseteq \mathcal{X}$ of streets defines a quiver $Q_{\mathcal{E}} = (\Lambda, \mathcal{E})$ and a homomorphism $\phi_{\mathcal{E}} \colon \mathbb{Q}Q_{\mathcal{E}} \to \Sigma(W)^{op}$.
- Pick a surjective one such that $\ker \phi_{\mathcal{E}}$ is admissible ...
- For Example A_3:



Solved and Open Problems.

Known:

- The quiver for type A_n [Garsia–Reutenauer, Schocker]
- The quiver for type B_n [Saliola]
- Quivers and relations for the exceptional types [Pf]

In Progress:

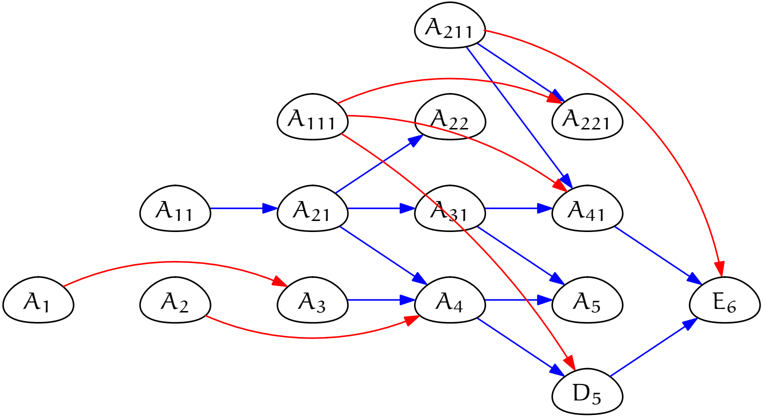
- Relations for type A_n [M. Bishop]

Open:

- Relations . . .
- Quiver for type D_n.

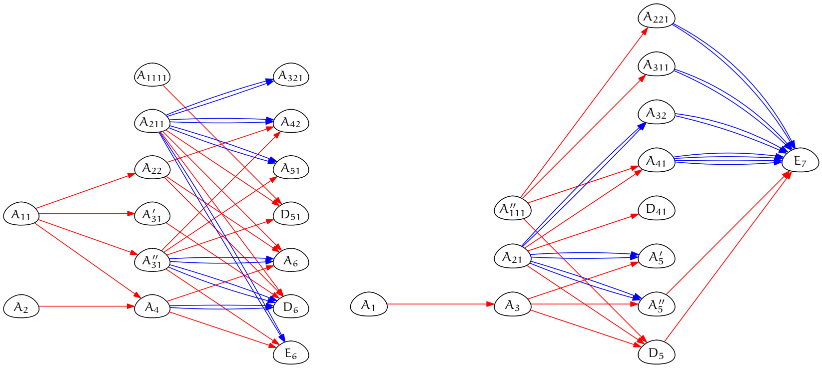
For Example E_6.

- The Quiver:

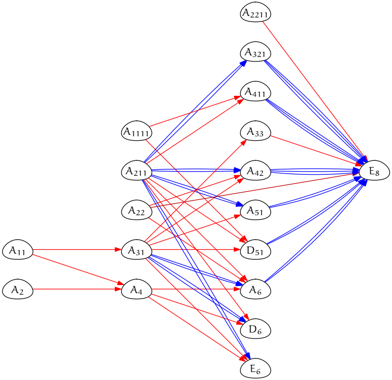


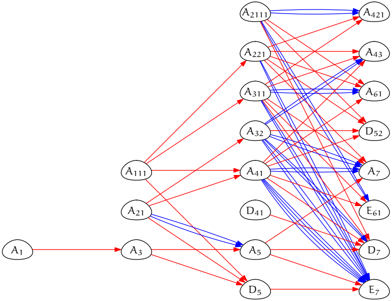
- Relations: $2(A_{111} \to A_{41} \to E_6) = (A_{111} \to D_5 \to E_6)$ and $(A_{11} \to A_{21} \to A_{31} \to A_5) + (A_{11} \to A_{21} \to A_4 \to A_5) = 0$.

For Example E_7.



For Example E_8.





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