Fundamental Lemma and Hitchin Fibration

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Introduction

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With Pierre-Henri Chaudouard, we have extended Ngô's proof in order to get Arthur's weighted Fundamental Lemma.

If time permits, I shall say a few words about our work at the end of the talk.

Arthur-Selberg trace formula

The Arthur-Selberg trace formula is a powerful tool for proving Langlands functorialities for automorphic representations:

- the cyclic base change,
- the transfer between inner forms,
- the transfer from classical groups to $\mathrm{GL}(n)$,
- and more generally all the Langlands functorialities of endoscopic type.

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Unfortunately, as it was first remarked by Langlands himself, in order to use that tool one first needs rather obscure and technical combinatorial identities between orbital integrals. These conjectural identities form the so-called Fundamental Lemma.

Fundamental Lemma

The Fundamental Lemma has been formulated by:

- Langlands and Shelstad (ordinary Fundamental Lemma),
- Kottwitz and Shelstad (twisted Fundamental Lemma),
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Several cases of the ordinary Fundamental Lemma have been proved by direct computations:

- $\mathrm{SL}(2)$ by Labesse-Langlands (1979),
- $\mathrm{U}(3)$ by Kottwitz (1992) and Rogawski (1990),
- $\mathrm{SL}(n)$ by Waldspurger (1991),
- $\mathrm{Sp}(4)$ by Hales (1997) and Weissauer (1993).

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- It is enough to consider a variant of the Fundamental Lemma for Lie algebras which is easier to formulate.
- The unequal characteristic case (over $\mathbb{Q}_p$) which is the most interesting, follows from the equal characteristic case (over $\mathbb{F}_p((t))$).
- The twisted versions of the Fundamental Lemma follow from the untwisted ones.

Geometric approaches

Using equivariant cohomology of affine Springer fibers, Goresky, Kottwitz and MacPherson have proved the Langlands-Shelstad Fundamental Lemma in the equivaluated unramified case.

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More recently, we have obtained the general case of the weighted Fundamental Lemma by extending Ngô's cohomological study to the hyperbolic part of the Hitchin fibration.

Notation

- k an algebraically closed field,
- G a semisimple algebraic group of adjoint type over k,
- $T \subset G$ a maximal torus,
- W the Weyl group of (G, T); we assume that $|W|$ is invertible in k,

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- $\mathfrak{g} \subset \mathfrak{t}$ the corresponging Lie algebras,
- n the rank of $\mathfrak{g}$, i.e. the dimension of $\mathfrak{t}$,
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- $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ the adjoint representation,
- C a smooth, connected, projective curve over k of genus g,
- $D = 2D' \subset C$ an even effective divisor of degree $> 2g$,
- ∞ a closed point of C which is not in the support of D.

Chevalley theorem

Chevalley has proved that

$$\mathfrak{car} := \mathfrak{t}//W = \mathrm{Spec}(k[\mathfrak{t}]^W) \cong \mathfrak{g}//G = \mathrm{Spec}(k[\mathfrak{g}]^G)$$

is a standard affine space of dimension n over k. Let $\chi : \mathfrak{g} \to \mathfrak{car}$ be the quotient morphism.

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Over C, we have:

- $\mathfrak{t}_D = \mathfrak{t} \otimes_k \mathcal{O}_C(D)$ and $\mathfrak{g}_D = \mathfrak{g} \otimes_k \mathcal{O}_C(D)$ viewed as vector bundles.
- W and G act in an obvious way on these vector bundles.
- The quotients are isomorphic affine bundles

$$\mathfrak{car}_D := \mathfrak{g}_D//G \cong \mathfrak{t}_D//W \to C.$$

- $\chi_D : \mathfrak{g}_D \twoheadrightarrow \mathfrak{car}_D$ the quotient morphism.
- χ_D makes sense when one replaces $\mathfrak{g}_D$ by $\mathrm{Ad}(\mathcal{E})(D)$ where $\mathcal{E}$ is any G-torsor on C.

Hitchin Fibration

Let $\mathcal{M}_G$ be the stack of triples $(\mathcal{E}, \theta, t_\infty)$ where:

- $\mathcal{E}$ is a G-torsor over C,
- $\theta \in H^0(C, \mathrm{Ad}(\mathcal{E})(D))$,
- $t_\infty \in \mathfrak{t}^{G-\mathrm{reg}}$ such that $\chi(t_\infty) = \chi(\theta_\infty) \in \mathfrak{car}$, where $\theta_\infty \in \mathrm{Ad}(\mathcal{E}_\infty) \cong \mathfrak{g}$ is the fiber of θ at ∞.

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Let $\mathcal{A}_G$ be the variety of pairs $a = (h, t_\infty)$ with $h \in \Gamma(\mathfrak{car}_D/C)$ and $t_\infty \in \mathfrak{t}^{G-\mathrm{reg}}$ such that $h(\infty) = \chi(t_\infty) \in \mathfrak{car}_{D,\infty} = \mathfrak{car}$.

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Definition

The Hitchin fibration is the stack morphism

$$f_G : \mathcal{M}_G \to \mathcal{A}_G$$

which maps $(\mathcal{E}, \theta, t_\infty)$ to $a = (h, t_\infty)$ where $h = \chi_D(\theta)$.

Basic properties of the Hitchin fibration

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By deformation theory it is easy to prove:

Proposition

The algebraic stack $\mathcal{M}_G$ is smooth over k.

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We have a closed embedding

$$i_M : \mathcal{A}_M^{G-\mathrm{reg}} \hookrightarrow \mathcal{A}_G$$

which maps (h_M, t_∞) to (h, t_∞) where

$$h : C \xrightarrow{h_M} \mathfrak{car}_{M,D} = \mathfrak{t}_D//W_M \twoheadrightarrow \mathfrak{t}_D//W_G = \mathfrak{car}_D.$$

The elliptic part

The elliptic part is the open subset

$$\mathcal{A}_G^{\mathrm{ell}} = \mathcal{A}_G - \bigcup_{\substack{M \in \mathcal{L}(T) \\ M \neq G}} \mathcal{A}_M^{G-\mathrm{reg}}.$$

Let $f_G^{\mathrm{ell}} = \mathcal{M}_G^{\mathrm{ell}} \to \mathcal{A}_G^{\mathrm{ell}}$ be the restriction of the Hitchin fibration to the elliptic open subset.

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The main geometric property of f_G^{ell} is:

Proposition (Faltings)

The algebraic stack $\mathcal{M}_G^{\mathrm{ell}}$ is a scheme (smooth over k) and the morphism f_G^{ell} is proper.

Adeles

Assume for a while that k is an algebraic closure of a finite field $\mathbb{F}_q$ and that C, D and ∞ are defined over $\mathbb{F}_q$. Then the set of $\mathbb{F}_q$-rational points of the Hitchin fibration has a nice adelic description.

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We shall use the standard notation:

- F the function field of C over $\mathbb{F}_q$,
- for each closed point c of C, $\mathcal{O}_c$ is the completed local ring of C at c and F_c its fraction field,
- $\mathbb{A}$ the ring of adeles of F,
- $\mathcal{O} = \prod_c \mathcal{O}_c \subset \mathbb{A}$ its maximal compact subring,
- for each closed point c in C, ϖ_c is an uniformizer of F_c,
- if $D = \sum_c d_c[c]$, $\varpi^{eD} = \prod_c \varpi_c^{ed_c}$ for any $e \in \mathbb{Z}$.

Adelic description of the Hitchin fibration

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Then at the level of $\mathbb{F}_q$-rational points, the Hitchin fibration is given by the map

$$\begin{array}{c} G(F)\backslash\{(g,\gamma)\in G(\mathbb{A})/G(\mathcal{O})\times\mathfrak{g}(F)\mid \mathrm{Ad}(g)^{-1}(\gamma)\in\varpi^{-D}\mathfrak{g}(\mathcal{O})\} \\ \downarrow \\ \{a\in F^r\mid \varpi^{e_iD}a_i\in\mathcal{O}\} \end{array}$$

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Here $(\chi_1,\dots,\chi_r)$ is a basis of the $\mathbb{F}_q$-algebras of G-invariant polynomials on $\mathfrak{g}$ and the e_i's are the degrees of the χ_i's.

Orbital integrals

Lemma

The number of $\mathbb{F}_q$-rational points of the Hitchin fiber $f_G^{-1}(a)$ at some point $a \in \mathcal{A}_G^{\mathrm{ell}}(\mathbb{F}_q)$ is the sum of the orbital integrals

$$J^G(\gamma_a, 1_{\varpi^{-D}\mathfrak{g}(\mathcal{O})}) = \int_{G_{\gamma_a}(F)\backslash G(\mathbb{A})} 1_{\varpi^{-D}\mathfrak{g}(\mathcal{O})}(\mathrm{Ad}(g)^{-1}(\gamma_a))\mathrm{d}g$$

where $\gamma_a \in \mathfrak{g}(F)$ runs through a set of representatives of the $G(F)$-conjugacy classes in $\chi^{-1}(a) \subset \mathfrak{g}(F)$.

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Here:

- $G_{\gamma_a} \subset G$ is the centralizer of γ_a,
- $1_{\varpi^{-D}\mathfrak{g}(\mathcal{O})}$ is the characteristic function of $\varpi^{-D}\mathfrak{g}(\mathcal{O})$ in $\mathfrak{g}(\mathbb{A})$,
- the Haar measure $\mathrm{d}g$ is normalized by $\mathrm{vol}(G(\mathcal{O}), \mathrm{d}g) = 1$.

Langlands duality

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Let $\hat{T} = \overline{\mathbb{Q}}_\ell^\times \otimes X^*(T)$ be the dual torus of T.

Let $(\hat{G}, \hat{T})$ be the Langlands dual of (G, T): $\hat{G}$ is a reductive group over $\overline{\mathbb{Q}}_\ell$ which admits $\hat{T}$ as a maximal torus and the roots of $\hat{T}$ in $\hat{G}$ are the coroots of T in G and vice versa.

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For example:

G	$\hat{G}$
$\mathrm{PSL}(n)$	$\mathrm{SL}(n)$
$\mathrm{PSp}(2n)$	$\mathrm{Spin}(2n+1)$
$\mathrm{SO}(2n+1)$	$\mathrm{Sp}(2n)$
$\mathrm{PO}(2n)$	$\mathrm{Spin}(2n)$

Endoscopic groups

For any $s \in \hat{T}$ we may consider the centralizer $Z_s(\hat{G})$ of s in $\hat{G}$:

- $Z_s(\hat{G})$ is a connected reductive group over $\overline{\mathbb{Q}}_\ell$ since $\hat{G}$ is simply connected (recall that G is adjoint).
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The endoscopic group H_s is said elliptic if $Z_s(\hat{G})$ is not contained in any proper Levi subgroup of $\hat{G}$.

Endoscopic strata

If $H = H_s$ is an endoscopic group, the Weyl group W^H of (H, T) is naturally embedded in W since it is equal to the Weyl group of $(Z_s(\hat{G}), \hat{T}) \subset (\hat{G}, \hat{T})$.

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Therefore, as for a Levi subgroup, we have a closed embedding

$$i_H : \mathcal{A}_H^{G-\mathrm{reg}} \hookrightarrow \mathcal{A}_G$$

which maps (h_H, t_∞) to (h, t_∞) where

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The endoscopic group H is elliptic if and only if

$$i_H^{-1}(\mathcal{A}_G^{\mathrm{ell}}) \neq \emptyset.$$

We denote by i_H^{ell} the restriction of i_H to the elliptic open subset.

Ngô's support theorem

We may consider the complex of ℓ-adic sheaves

$$Rf^{\mathrm{ell}}_{G,*}\overline{\mathbb{Q}}_\ell$$

over $\mathcal{A}_G$ and the direct sum of its perverse cohomology sheaves

$$\bigoplus_i {}^{\mathrm{p}}\mathcal{H}^i(Rf^{\mathrm{ell}}_{G,*}\overline{\mathbb{Q}}_\ell)$$

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Theorem (Ngô Bao Châu)

The support of any irreducible constituent of $\oplus_i {}^{\mathrm{p}}\mathcal{H}^i(Rf_{G,*}^{\mathrm{ell}}\overline{\mathbb{Q}}_\ell)$ is equal to $\mathrm{A}_G^{\mathrm{ell}} \cap i_H(\mathcal{A}_H^{G-\mathrm{reg}})$ for some elliptic endoscopic group H.

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- In positive characteristic, it might be also true in general but for the moment it is only proved over a dense open subset of $\mathcal{A}_G$, the so-called good open subset.
- Fortunately, the good open subset is large enough for proving the (local) Fundamental Lemma.

Symmetries

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- As θ_∞ is semisimple, the restriction of $(\mathcal{E}, \theta)$ to $\mathrm{Spf}(\mathcal{O}_\infty)$ admits a reduction to T.
- This reduction is canonical if one takes into account the extra data t_∞.
- Therefore, the set of glueing data can be identified with

$$X_*(T) \cong T(F_\infty)/T(\mathcal{O}_\infty).$$

Symmetries

The Hitchin fibration has a lot of symmetries. In particular, it is equipped with an action of the discrete group $X_*(T)$ which can be described as follows:

- A Hitchin pair $(\mathcal{E}, \theta)$ on C can be obtained by gluing a Hitchin pair on $C - \{\infty\}$ and a Hitchin pair on the formal completion $\mathrm{Spf}(\mathcal{O}_\infty)$ of C at ∞.
- As θ_∞ is semisimple, the restriction of $(\mathcal{E}, \theta)$ to $\mathrm{Spf}(\mathcal{O}_\infty)$ admits a reduction to T.
- This reduction is canonical if one takes into account the extra data t_∞.
- Therefore, the set of glueing data can be identified with

$$X_*(T) \cong T(F_\infty)/T(\mathcal{O}_\infty).$$

- Hence we have an action by translation of $X_*(T)$ on the set of gluing data.

Ngô's main theorem

Over $\mathcal{A}_G^{\mathrm{ell}}$ the action of $X_*(T)$ is of finite order and thus we have a canonical decomposition into isotypical components

$$^{\mathrm{p}}\mathcal{H}^\bullet(Rf_{G,*}^{\mathrm{ell}}\overline{\mathbb{Q}}_\ell) = \bigoplus_{s\in\hat{T}^\mu} {}^{\mathrm{p}}\mathcal{H}^\bullet(Rf_{G,*}^{\mathrm{ell}}\overline{\mathbb{Q}}_\ell)_s$$

where $\hat{T}^\mu \subset \hat{T} = \mathrm{Hom}(X_*(T), \overline{\mathbb{Q}}_\ell^\times)$ is the torsion subgroup.

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Theorem (Ngô)

Let $s \in \hat{T}^\mu$. If H_s is elliptic, we have

$$ {}^{\mathrm{p}}\mathcal{H}^\bullet(Rf^{\mathrm{ell}}_{G,*}\overline{\mathbb{Q}}_\ell)_s = (i^{\mathrm{ell}}_{H_s})_* {}^{\mathrm{p}}\mathcal{H}^{\bullet-2r_{H_s}}(Rf^{\mathrm{ell}}_{H_s,*}\overline{\mathbb{Q}}_\ell)_1(-r_{H_s}) $$

where r_{H_s} is the codimension of $\mathcal{A}_{H_s}^{G-\mathrm{reg}}$ in $\mathcal{A}_G$. Otherwise, we have

$$ {}^{\mathrm{p}}\mathcal{H}^\bullet(Rf^{\mathrm{ell}}_{G,*}\overline{\mathbb{Q}}_\ell)_s = (0). $$

Fundamental Lemma

If we assume again that k is an algebraic closure of a finite field $\mathbb{F}_q$ and that C, D and ∞ are defined over $\mathbb{F}_q$, we can take the traces of Frobenius on both sides of Ngô's main theorem.

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Finally, a standard global-to-local argument gives the Langlands-Shelstad Fundamental Lemma for Lie algebras in the equal characteristic case.

Chaudouard-L.'s extension of Ngô's work

Outside the elliptic locus, the Hitchin fibration is neither of finite type nor separated. One first needs to truncate it. We do it by using Mumford stability.

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In some sense the weighted Fundamental Lemma follows from the ordinary Fundamental Lemma by middle extension.