Types for *p*-adic classical groups

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21 May 2009
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Notations

- F a locally compact non-archimedean local field,
  complete with respect to $|\cdot|$ $\qquad\mathbb{Q}_p, \quad \mathbb{F}_q((\varpi))$
- $\mathfrak{o}_F = \overline{B}(0,1)$, ring of integers, $\qquad\mathbb{Z}_p, \quad \mathbb{F}_q[[\varpi]]$
- $\mathfrak{p}_F = B(0,1)$, unique maximal ideal, $\qquad p\mathbb{Z}_p, \varpi\mathbb{F}_q[[\varpi]]$
  principal, with generator ϖ (a *uniformizer*)
- $k_F = \mathfrak{o}_F/\mathfrak{p}_F \simeq \mathbb{F}_q$ the residue field, $\qquad\mathbb{F}_p, \quad \mathbb{F}_q$
  of characteristic p.

$\{\mathfrak{p}_F^n : n \in \mathbb{Z}\}$ is a fundamental system of neighbourhoods of 0,

$$\bigcap_{n\in\mathbb{Z}} \mathfrak{p}_F^n = \{0\}, \qquad \bigcup_{n\in\mathbb{Z}} \mathfrak{p}_F^n = F.$$

Similarly, $\quad F^\times \supset \mathfrak{o}_F^\times \supset 1 + \mathfrak{p}_F^n, \quad$ for $n \geq 1$.

Notations

- G the F-points of a connected reductive group over F.

$$\boldsymbol{GL_n}(F),$$

$$\boldsymbol{Sp_{2n}}(F) = \{g \in \boldsymbol{GL_{2n}}(F) : J^{-1} g^T J = g^{-1}\},$$

where

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}.$$

Compact open subgroups

Maximal compact open subgroups in $\boldsymbol{GL}_n(F)$ are all conjugate to

$$K = \boldsymbol{GL}_n(\mathfrak{o}_F),$$

with pro-p radical

$$K^1 = I_n + \mathbb{M}_n(\mathfrak{p}_F),$$

and filtration by normal subgroups

$$K^m = I_n + \mathbb{M}_n(\mathfrak{p}_F^m), \quad m \geq 1,$$

a fundamental system of neighbourhoods of the identity I_n.

Note that $K/K^1 \simeq \boldsymbol{GL}_n(\mathbb{F}_q)$.

Compact open subgroups

Maximal compact open subgroups in $\boldsymbol{Sp}_{2n}(F)$ are conjugate to

$$K = \begin{array}{c} \left(\begin{array}{c|c|c} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{p}_F^{-1} \\ \hline \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \hline \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \end{array}\right) \\ \begin{array}{ccc} \underbrace{\qquad}_{r} & \underbrace{\qquad\quad}_{2(n-r)} & \underbrace{\qquad}_{r} \end{array} \end{array} \cap G, \qquad 0 \le r \le n,$$

with pro-p radical $K^1 = I_{2n} + \left(\begin{array}{c|c|c} \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \hline \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \\ \hline \mathfrak{p}_F^2 & \mathfrak{p}_F & \mathfrak{p}_F \end{array}\right) \cap G,$

and filtration by normal subgroups K^m, for $m \ge 1$.

In this case $K/K^1 \simeq \boldsymbol{Sp}_{2r}(\mathbb{F}_q) \times \boldsymbol{Sp}_{2(n-r)}(\mathbb{F}_q)$.

Representations

- $\mathfrak{R}(G)$ the category of smooth complex representations $\pi : G \to \mathrm{Aut}_{\mathbb{C}}\mathcal{V}$.

  A complex representation $(\pi, \mathcal{V})$ of G is *smooth* if

  $$\mathrm{Stab}_G(\boldsymbol{v}) \text{ is open, for all } \boldsymbol{v} \in \mathcal{V}.$$

- $\mathrm{Irr}\,(G)$ the set of equivalence classes of irreducible representations in $\mathfrak{R}(G)$.

Facts

- An irreducible smooth representation is either
  - 1-dimensional – a (quasi-)character; or
  - countably infinite-dimensional.

  An unramified character of G is a character trivial on every compact open subgroup.
  For $\boldsymbol{GL}_n(F)$ the unramified characters take the form
  $$|\det(\cdot)|^s, \qquad \text{for } s \in \mathbb{C}.$$
- If $\pi \in \operatorname{Irr}(G)$ then $\pi|_K$ decomposes as a direct sum of $\sigma \in \operatorname{Irr}(K)$, each with finite multiplicity.

  Moreover, since σ is smooth and finite-dimensional, $\ker(\sigma) \supset K^m$, for large enough m, and σ factors through the finite group K/K^m.
- We have Schur's Lemma but not Maschke's Theorem.

Parabolic subgroups

- $P = MN$ denotes a parabolic subgroup of G, with unipotent radical N and Levi subgroup M.

  For $\boldsymbol{GL}_n(F)$ or $\boldsymbol{Sp}_{2n}(F)$,

  - P is (conjugate to) block upper triangular;
  - N is the subgroup with identity diagonal blocks;
  - M is (conjugate to) block diagonal.

For $\boldsymbol{GL}_n(F)$, $\quad M \cong \prod_{i=1}^r \boldsymbol{GL}_{n_i}(F), \qquad \sum_{i=1}^r n_i = n$.

For $\boldsymbol{Sp}_{2n}(F)$, $\quad M \cong \boldsymbol{Sp}_{2n_0}(F) \times \prod_{i=1}^r \boldsymbol{GL}_{n_i}(F), \quad \sum_{i=0}^r n_i = n$.

Parabolic induction

If M is a Levi subgroup of G and $\tau : M \to \mathrm{Aut}_{\mathbb{C}}\mathcal{W}$ is an irreducible representation, we can consider τ as a representation of P by inflation via $M \simeq P/N$, and form the induced representation

$$\mathrm{Ind}\,_{M,P}^{G}\,\tau,$$

the space of functions $f : G \to \mathcal{W}$ such that

- $f(pg) = \delta_P(p)^{1/2}\tau(p)f(g)$, for all $p \in P$, $g \in G$,
- there is a compact open subgroup K of G such that $f(gk) = f(g)$ for all $g \in G$, $k \in K$,
- supp.f is compact modulo P,

with right regular action. It is a smooth representation of finite length, whose composition factors depend only on (M, τ), not on P.

Parabolic induction

Example $G = \boldsymbol{GL}_2(F)$.

If $M = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and $\tau = \chi = \chi_1 \otimes \chi_2$ then

- $\mathrm{Ind}_{M,P}^{G}\,\chi$ is irreducible unless $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$;
- if $\chi_1\chi_2^{-1} = |\cdot|^{-1}$ then $\mathrm{Ind}_{M,P}^{G}\,\chi$ has a unique 1-dimensional subrepresentation, with irreducible quotient;
- if $\chi_1\chi_2^{-1} = |\cdot|$ then $\mathrm{Ind}_{M,P}^{G}\,\chi$ has a unique irreducible subrepresentation, with 1-dimensional quotient.

The irreducible representations of G obtained in this way (with P the Borel subgroup as above) are called *principal series* representations.

Cuspidal representations

An irreducible representation π of G is cuspidal if it is *not* a subrepresentation of any parabolically induced representation $\mathrm{Ind}\,_{M,P}^{G}\,\tau$, for M a *proper* Levi subgroup.

Theorem (Jacquet)

Given $\pi \in \mathrm{Irr}\,(G)$*, there are a Levi subgroup* M *and a cuspidal representation* τ *of* M *such that*

$$\pi \hookrightarrow \mathrm{Ind}\,_{M,P}^{G}\tau.$$

Moreover, (M, τ) *is unique up to conjugacy. It is called the cuspidal support of* π*.*

Inertial equivalence

We define inertial equivalence on the set of pairs (M, τ) as above by:

$$(M, \tau) \sim (M', \tau') \Leftrightarrow \begin{cases} \text{there are } g \in G \text{ and an} \\ \text{unramified character } \chi \text{ of } M \text{ s.t.} \\ {}^{g}M' = M \ \text{ and } \ {}^{g}\tau' \simeq \tau \otimes \chi. \end{cases}$$

Write $[M, \tau]_G$ for the inertial equivalence class of (M, τ) and $\mathfrak{B}(G)$ for the set of all inertial equivalence classes.

For $\mathfrak{s} \in \mathfrak{B}(G)$, write $\mathfrak{R}^{\mathfrak{s}}(G)$ for the full subcategory of $\mathfrak{R}(G)$ of representations all of whose irreducible subquotients have cuspidal support in $\mathfrak{s}$.

Bernstein decomposition

Theorem (Bernstein)

The category $\mathfrak{R}(G)$ *decomposes as a direct product*

$$\mathfrak{R}(G) \;=\; \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

That is, for $\pi : G \to \mathrm{Aut}_{\mathbb{C}}\mathcal{V}$ and $\tau : G \to \mathrm{Aut}_{\mathbb{C}}\mathcal{W}$, we have

$$\mathcal{V} = \bigoplus_{\mathfrak{s}} \mathcal{V}^{\mathfrak{s}}, \text{ with } \mathcal{V}^{\mathfrak{s}} \in \mathfrak{R}^{\mathfrak{s}}(G),$$

$$\mathrm{Hom}_{\,G}(\mathcal{V},\mathcal{W}) = \prod_{\mathfrak{s}} \mathrm{Hom}_{\,G}(\mathcal{V}^{\mathfrak{s}},\mathcal{W}^{\mathfrak{s}})$$

Types

To understand the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, we use Bushnell–Kutzko's *theory of types*.

The idea is to identify $\mathfrak{R}^{\mathfrak{s}}(G)$ as the category of modules over a Hecke algebra in the following way:

> *Find a pair* (K, ρ)*, with* K *a compact open subgroup and* $\rho \in \mathrm{Irr}\,(K)$*, such that, for* $\pi \in \mathrm{Irr}\,(G)$*:*
>
> $$\pi \in \mathfrak{R}^{\mathfrak{s}}(G) \Leftrightarrow \mathrm{Hom}\,_K(\pi, \rho) \neq 0.$$
>
> *Now put* $\mathcal{H}(G, \rho) = \mathrm{End}_G(\mathrm{Ind}\,_K^G(\rho))$ *and we have an equivalence of categories*
>
> $$\mathfrak{R}^{\mathfrak{s}}(G) \;\longleftrightarrow\; \mathcal{H}(G, \rho)\text{-}Mod,$$

Such a pair (K, ρ) is called an $\mathfrak{s}$-type.

Cuspidal types: $\mathfrak{s} = [G, \pi]_G$

All known cuspidal representations can be constructed in the following way:

$$\pi \simeq \mathrm{Ind}\,_{\widetilde{K}}^{G}\,\widetilde{\rho},$$

for $\widetilde{K}$ a compact-mod-center open subgroup of G and $\widetilde{\rho} \in \mathrm{Irr}\,(\widetilde{K})$, such that, for K the unique maximal compact subgroup of $\widetilde{K}$, the restriction $\rho = \widetilde{\rho}|_K$ is irreducible. Then (K, ρ) is an $\mathfrak{s}$-type.

Cuspidal types: $\mathfrak{s} = [G, \pi]_G$

Theorem

This is all cuspidal representations in the following cases:

- ***GL***$_n(F)$ *[Bushnell–Kutzko, '91]*
- ***SL***$_n(F)$ *[Bushnell–Kutzko, '93]*
- *Level zero representations for arbitrary G* *[Morris, Moy–Prasad, '96]*
- *G arbitrary but several conditions on F* *[Kim–Yu, '06]*
- *G classical,* $p \neq 2$ *[S. '06]*
- ***GL***$_m(D)$, *D division algebra* *[Sécherre–S., '06]*

Cuspidal types: $\mathfrak{s} = [G, \pi]_G$

All constructions are broadly similar: we get

- J a special compact open subgroup of G;
- κ a very special representations of J;
- J/J^1 is isomorphic to a product of reductive groups over finite fields, and we take σ an irreducible cuspidal representation of J/J^1.

Put $\lambda = \kappa \otimes \sigma$ and then (J, λ) is a cuspidal type.

In the level zero case, we have $J = K$ a maximal compact open subgroup of G and $\kappa = \mathbf{1}$, the trivial representation.

Non-cuspidal types: $\mathfrak{s} = [M, \tau]_G$

Put $\mathfrak{s}_M = [M, \tau]_M$ and suppose we have a (cuspidal) $\mathfrak{s}_M$-type (K_M, ρ_M).

Let $P = MN$ be a parabolic, with opposite parabolic $P_- = MN_-$.

The idea is to use (K_M, ρ_M) to produce a pair (K, ρ) with the following properties:

- $K \cap M = K_M$ and $\quad K = (K \cap N_-)K_M(K \cap N)$;
- $\rho(k_- k_M k_+) = \rho_M(k_M)$;
- $\mathcal{H}(G, \rho)$ contains an invertible function supported on the (K, K)-double coset of an element $\zeta \in Z(M)$ which contracts N and expands N_-:

$$\bigcap_{n \geq 0} \zeta^n (K \cap N) \zeta^{-n} = \{1\}, \text{ and } \bigcap_{n \leq 0} \zeta^n (K \cap N_-) \zeta^{-n} = \{1\}.$$

Non-cuspidal types: $\mathfrak{s} = [M, \tau]_G$

Theorem (Bushnell–Kutzko)

In this situation (K, ρ) *is an* $\mathfrak{s}$*-type (called a cover of* (K_M, ρ_M)*) and there is an algebra embedding*

$$t_P : \mathcal{H}(M, \rho_M) \hookrightarrow \mathcal{H}(G, \rho)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{R}^{\mathfrak{s}}(G) \arrow[r, leftrightarrow] & \mathcal{H}(G, \rho)\text{-}Mod \\
\mathfrak{R}^{\mathfrak{s}_M}(M) \arrow[r, leftrightarrow] \arrow[u, "\mathrm{Ind}\,_{M,P}^{G}"] & \mathcal{H}(M, \rho_M)\text{-}Mod \arrow[u, "(t_P)_*"']
\end{tikzcd}$$

In particular, knowledge of the Hecke algebras and the embedding t_P determines the reducibilities of $\mathrm{Ind}\,_{M,P}^{G}$.

An example in $\boldsymbol{Sp}_{2n+2}(F)$

Take $M \simeq \boldsymbol{GL}_n(F) \times \boldsymbol{SL}_2(F)$.

If $\tau \in \mathrm{Irr}\,(\mathfrak{R}^{\mathfrak{s}_M}(M))$ then

$$\tau \simeq \left(\tau_0 \otimes |\det(\cdot)|^s\right) \otimes \tau',$$

with τ_0 a cuspidal representation of $\boldsymbol{GL}_n(F)$ and τ' a cuspidal representation of $\boldsymbol{SL}_2(F)$.

Question For what values of $s \in \mathbb{C}$ is $\mathrm{Ind}\,_{M,P}^{G}\tau$ irreducible?

- If no $\tau_0 \otimes |\det(\cdot)|^s$ is self-dual then there are no such s.
- If τ_0 is self-dual then one can compute the values from the parameters in Hecke algebras over finite fields.

An example in $Sp_{2n+2}(F)$

Idea: (J_M, λ_M) the type for $[M, \tau]_M$, with $\lambda_M = \kappa_M \otimes \sigma_M$.

(J_P, λ_P) the cover, with $\lambda_P = \kappa_P \otimes \sigma_M$

We can find compact open $J \supset J_P$ such that:

- J/J^1 is a reductive group over a finite field;
- J_P/J^1 is a proper parabolic subgroup, with Levi J_M/J_M^1.

Then we get a support-preserving algebra embedding:

$$\mathcal{H}(J/J^1, \sigma_M) \hookrightarrow \mathcal{H}(G, \lambda_P).$$

An example in $\boldsymbol{Sp}_{2n+2}(F)$

In the case of level zero representations, we have:

$$J_M \simeq \boldsymbol{GL}_n(\mathfrak{o}_F) \times \boldsymbol{SL}_2(\mathfrak{o}_F), \qquad \lambda_m = \sigma_M = \sigma_0 \otimes \sigma',$$

with σ_0, σ' inflations of cuspidal representations.

Because τ_0 is self-dual, so is σ_0 so that either

- $n = 1$ and $\sigma_0 = \mathbf{1}$ or the quadratic character; or
- $n = 2m$ is even.

An example in $\boldsymbol{Sp}_{2n+2}(F)$

Then

$$J_P = \left(\begin{array}{c|c|c} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \hline \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \hline \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \end{array}\right) \cap G,$$

which is contained in

$$J_a = \boldsymbol{Sp}_{2n+2}(\mathfrak{o}_F) \quad \text{and} \quad J_b = \left(\begin{array}{c|c|c} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{p}_F^{-1} \\ \hline \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \hline \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F \end{array}\right) \cap G.$$

Then $J_a/J_a^1 \simeq \boldsymbol{Sp}_{2n+2}(\mathbb{F}_q)$ and $J_b/J_b^1 \simeq \boldsymbol{Sp}_{2n}(\mathbb{F}_q) \times \boldsymbol{SL}_2(\mathbb{F}_q)$.

An example in $\boldsymbol{Sp}_{2n+2}(F)$

Put $$w_a = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \qquad w_b = \begin{pmatrix} & & \varpi^{-1} \\ & 1 & \\ \varpi & & \end{pmatrix}.$$

In the case $n = 2m$:

$\mathcal{H}(J_b/J_b^1, \sigma_M)$ is 2-dimensional, with generator T_b supported on w_b such that

$$(T_b - q^m)(T_b + 1) = 0.$$

If $n > 2$ then $\mathcal{H}(J_a/J_a^1, \sigma_M)$ is 2-dimensional, with generator T_a supported on w_a such that

$$(T_a - q^m)(T_a + 1) = 0.$$

An example in $\boldsymbol{Sp}_{2n+2}(F)$

Put $\quad w_a = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \qquad w_b = \begin{pmatrix} & & \varpi^{-1} \\ & 1 & \\ \varpi & & \end{pmatrix}.$

In the case $n = 2m$:

$\mathcal{H}(J_b/J_b^1, \sigma_M)$ is 2-dimensional, with generator T_b supported on w_b such that

$$(T_b - q^m)(T_b + 1) = 0.$$

If $n = 2$ it depends on a compatibility condition between σ_0 and σ': we get

$$(T_a - q^3)(T_a + 1) = 0 \qquad \text{or} \qquad (T_a - q)(T_a + 1) = 0.$$

An example in $\mathbf{Sp}_{2n+2}(F)$

If $n = 1$:

$\mathcal{H}(J_b/J_b^1, \sigma_M)$ has parameter $\begin{cases} q & \text{if } \sigma_0 = \mathbf{1}, \\ 1 & \text{if } \sigma_0 \neq \mathbf{1}. \end{cases}$

$\mathcal{H}(J_a/J_a^1, \sigma_M)$ has parameter

$$\begin{cases} q^2 & \text{if } \sigma_0 \neq \mathbf{1} \text{ and } \dim \sigma' = \frac{1}{2}(q-1), \\ q & \text{otherwise.} \end{cases}$$

An example in $\mathbf{Sp}_{2n+2}(F)$

Now, if v_a, v_b denote the exponents of q in the two cases, we get reducibility of parabolic induction at

$$s = \pm\left(\frac{v_a+v_b}{2n}\right), \pm\left(\frac{v_a-v_b}{2n}\right) + \frac{\pi i}{n\log q}.$$

For the Langlands correspondence, one is interested in reducibilities with $|\Re(s)| \geq 1$ – these determine the correspondence for $\mathbf{SL}_2(F)$ explicitly.

- For any σ' with $\sigma_0 = \mathbf{1}$ we get $s = \pm 1$.
- For $\dim\sigma' = \frac{1}{2}(q-1)$ with σ_0 the quadratic character we get $s = \pm 1, \pm 1 + \frac{\pi i}{\log q}$.
- for $\dim\sigma' = q-1$ with σ_0 the compatible cuspidal of $\mathbf{GL}_2(\mathbb{F}_q)$ we get $s = \pm 1$.

An example in $\boldsymbol{Sp}_{2n+2}(F)$

For non-level zero representations we get something similar; we end up looking at

$$U(2t+1) \supset \boldsymbol{GL}_t \times U(1), \qquad O(2t+1) \supset \boldsymbol{GL}_t \times \{\pm 1\}.$$

In current work with Blondel and Henniart, we are doing this with $\boldsymbol{Sp}_4(F)$ in place of $\boldsymbol{SL}_2(F)$.

e.g. For $\mathbf{1} \otimes \theta_{10}$ on $\boldsymbol{GL}_1(\mathbb{F}_q) \times \boldsymbol{Sp}_4(\mathbb{F}_q)$, we get reducibility at $s = \pm 2, \pm 1 + \frac{\pi i}{\log q}$.

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