Atypicality, Complexity and Module Varieties for Classical Lie Superalgebras

Daniel K. Nakano

University of Georgia

Workshop on Representation Theory and Lie Theory
Newton Institute
Cambridge University, UK

joint work with Brian Boe and Jonathan Kujawa

References

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Motivation

Let us first consider blocks of the Category $\mathcal{O}$ (or relative Category $\mathcal{O}$) for complex semisimple Lie algebras:

- *these are highest weight categories with finitely many simple modules,*
- *the projective resolutions of modules have finite length,*
- *the cohomology only lives in finitely many degrees.*

Furthermore, if one takes the endomorphism algebra of a progenerator this gives rise to a finite-dimensional quasi hereditary algebra (which is Morita equivalent to $\mathcal{O}$).

Motivation

On the other hand, if one considers the category $\mathcal{F}$ of finite-dimensional modules for a classical Lie superalgebra (i.e., $\mathfrak{g} = \mathfrak{g}l(m|n)$) which are completely reducible over $\mathfrak{g}_{\bar{0}}$ then

- *this is also a highest weight category (as observed by Brundan),*
- *there are infinitely many simple modules,*
- *$\mathcal{F}$ is self-injective (i.e., projective is equivalent to injective)* [BKN3],
- *projective resolutions have infinite length, and the terms can grow in dimension.*

Thus, the cohomology can also grow in dimension so one is motivated to study these objects with ideas and tools from modular representation theory.

Motivation

It is useful to think about modeling our approach on finite group schemes or finite-dimensional cocommutative Hopf algebras. Let A be a finite dimensional cocommutative Hopf algebra over an algebraically closed field k.

• *A is a Frobenius algebra (an A-module is projective if and only if it is injective).*

• *every finite dimensional A-module admits a minimal projective resolution whose terms have dimensions which increase at a polynomial rate of growth.*

Motivation

Observe that if A is a finite-dimensional algebra which is Frobenius and quasi-hereditary then A is semisimple.

Therefore, in order to have both features (i.e., Frobenius and highest weight category) and have a non-trivial theory, we must have infinitely many simple modules in our module category.

Throughout let $k = \mathbb{C}$. Let $\mathfrak{g}$ be a *Lie superalgebra* which is a $\mathbb{Z}_2$-graded vector space

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with a bracket operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which preserves the $\mathbb{Z}_2$-grading and satisfies graded versions of the usual Lie bracket axioms.

Definition

A finite dimensional Lie superalgebra $\mathfrak{g}$ is called *classical* if there is a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$ and an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.

The Lie Superalgebra $\mathfrak{gl}(m|n)$

Example

The underlying vector space for $\mathfrak{gl}(m|n)$ is the set of $m \times n$ matrices over $\mathbb{C}$. We have $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, where $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Moreover, $\mathfrak{g}_{\bar{1}}$ consists of matrices

$$\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.$$

The supercommutator is given by

$$[E_{i,j}, E_{k,l}] = E_{i,j}E_{k,l} - (-1)^{\bar{E}_{i,j}\bar{E}_{k,l}} E_{k,l}E_{i,j}.$$

Let $\mathcal{F}$ be the full subcategory of $\mathcal{C}_{(\mathfrak{g},\mathfrak{g}_0)}$ (relative category) where the objects are the finite dimensional supermodules which are completely reducible as $\mathfrak{g}_0$-supermodules.

$U(\mathfrak{g})$ universal enveloping algebra of $\mathfrak{g}$

$S^\bullet(\mathfrak{g}_{\bar{1}}^*)$ symmetric algebra on the dual of $\mathfrak{g}_{\bar{1}}$.

Definition

A *basic classical* Lie superalgebra is a classical Lie superalgebra with a nondegenerate invariant supersymmetric even bilinear form.

For $\mathfrak{g}$ basic classical:

Δ the set of roots; Δ_0 the set of even roots; Δ_1 the set of odd roots.

$\rho = \frac{1}{2}(\sum_{\alpha\in\Delta_0^+} \alpha - \sum_{\alpha\in\Delta_1^+} \alpha)$.

Defect and Atypicality

Definition

(Kac-Wakimoto) Let $\mathfrak{g}$ be a basic classical Lie algebra. The *defect* of $\mathfrak{g}$, denoted by $\text{defect}(\mathfrak{g})$, is the dimension of the maximal isotropic subspace in the $\mathbb{R}$-span of Δ.

Example

- $\text{defect}(\mathfrak{gl}(m|n)) = \text{defect}(\mathfrak{sl}(m|n)) = \min(m, n)$,
- $\text{defect}(D(2, 1; \alpha)) = \text{defect}(G(3)) = 1$.

Definition

Let $\mathfrak{g}$ be a basic classical Lie algebra and $\lambda \in \mathfrak{t}^*$ be a weight. The *atypicality* of λ, denoted by $\text{atp}(\lambda)$, is the maximal number of linearly independent mutually orthogonal, positive isotropic roots $\alpha \in \Delta^+$ such that $(\lambda + \rho, \alpha) = 0$.

Example

Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. The simple modules in the principal block are one dimensional and indexed by $L(\lambda \,|\, -\lambda)$ where $\lambda \in \mathbb{Z}$. The projective cover $P(\lambda \,|\, -\lambda)$ of $L(\lambda \,|\, -\lambda)$ is four dimensional.

The minimal projective resolution of the trivial module $L(0 \,|\, 0)$ is given by

$$\cdots \to P(1 \,|\, -1) \oplus P(-1 \,|\, 1) \to P(0 \,|\, 0) \to L(0 \,|\, 0) \to 0.$$

Therefore, $\dim P_n = 4(n+1)$ and $c_{\mathcal{F}}(L(0 \,|\, 0)) = 2$ (rate of growth of the minimal proj. resolution).

In fact, one can easily show that $c_{\mathcal{F}}(L(\lambda \,|\, -\lambda)) = 2$ for all $\lambda \in \mathbb{Z}$. The atypicality of every simple module in the prinicipal block is one and equal to $\dim \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$.

Relative Cohomology and Detecting Subalgebras

Let $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, M)$ be the relative Lie algebra cohomology of the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ which is obtained from the complex

$$C^\bullet = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\Lambda^\bullet_{super}(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M).$$

Theorem

Let $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ be as above. Then

$$\mathrm{Ext}^\bullet_{\mathcal{F}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong (\Lambda^\bullet_{super}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*)^{G_{\bar{0}}} \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}.$$

Note that the cohomology ring is finitely generated because $G_{\bar{0}}$ is reductive.

Theorem

Let $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ *be a classical (simple) Lie superalgebra. Then there exists a subalgebra* $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ *isomorphic to* $\oplus \mathfrak{q}(1)$ *and a finite reflection group* W *such that restriction map*

$$\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \to \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^W.$$

is an isomorphism.

Theorem

Let $\mathfrak{g}$ *be a basic classical simple Lie superalgebra. Moreover, assume that* $\mathfrak{g} \not\cong \mathfrak{psl}(n|n)$*. Then the following are equal:*

(a) $\mathrm{defect}(\mathfrak{g})$

(b) $\dim \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$;

(c) $\dim \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})$.

(d) $\dim \mathfrak{e}_{\bar{1}}$.

Support Varieties and Atypicality

Using the finite generation of cohomology [BKN1] we can define the following support varieties for modules over $\mathfrak{g}$ and $\mathfrak{e}$:

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \text{Maxspec}(\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}};\mathbb{C})/J_{\mathfrak{g}}(M \otimes M^{*})) \subseteq \mathfrak{g}_{\bar{1}}/G_{\bar{0}}$$

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(N) = \text{Maxspec}(\mathrm{H}^{\bullet}(\mathfrak{e},\mathfrak{e}_{\bar{0}};\mathbb{C})/J_{\mathfrak{e}}(N \otimes N^{*})) \subseteq \mathfrak{e}_{\bar{1}}$$

where $J_{\mathfrak{g}}(M \otimes M^{*})$ is the annihilator of the cohomology ring on $\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}}; M \otimes M^{*})$. The ideal $J_{\mathfrak{e}}(-)$ is similarly defined.

Let M be a supermodule for $\mathfrak{g}$. The restriction map in cohomology induces a map res^{*} on these varieties with

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)/W \hookrightarrow \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M).$$

Atypicality Conjecture

Conjecture

Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Then

$$\mathrm{atyp}(\lambda) = \dim \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda)).$$

Note that if this conjecture is true then one can extend the definition of atypicality to all modules in $\mathcal{F}$.

Theorem

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, $r = \mathrm{defect}(\mathfrak{g})$, *and* $L(\lambda)$ *be a simple* $\mathfrak{g}$*-module of atypicality* k*. Then*

(a)

$$\mathrm{res}^* \left(\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}\left(L(\lambda)\right) \right) = \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}\left(L(\lambda)\right) \cong \mathbb{A}^k.$$

(b)

$$\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda)) = W \cdot \widetilde{\mathfrak{e}}_{\bar{1}}.$$

where $\dim \widetilde{\mathfrak{e}}_{\bar{1}} = k$*. In particular,* $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda))$ *is the union of* $\binom{r}{k}$ k*-dimensional subspaces.*

Complexity for modules in $\mathcal{F}$

Definition

Let $\mathcal{V} = \{V_t : \ t \in \mathbb{N}\} = \{V_\bullet\}$ be a sequence of finite dimensional $\mathbb{C}$-vector spaces. The *rate of growth* of $\mathcal{V}$, $r(\mathcal{V})$, is the smallest positive integer c such that $\dim_k V_t \leq C \cdot t^{c-1}$ for some constant $C > 0$. If no such integer exists then $\mathcal{V}$ has infinite rate of growth.

Definition

Let $M \in \mathcal{F}$ and $P_\bullet \twoheadrightarrow M$ be a minimal projective resolution for M. Following Alperin (1977), we define the *complexity of M* to be $r(\{P_n : \ n = 0, 1, 2, \ldots\})$.

Theorem

Let M be an object of $\mathcal{F}$. Then

(i) $c_{\mathcal{F}}(M) = 0$ *if and only if M is projective;*

(ii) $c_{\mathcal{F}}(M) \leq \dim \mathfrak{g}_{\bar{1}}$.

Part (i) follows by the self injectivity of $\mathcal{F}$. For part (ii) we construct an explicit projective resolution for the trivial module

$$\cdots \xrightarrow{\partial_3} D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} D_0 \xrightarrow{\partial_0} \mathbb{C} \to 0$$

where

$$D_p := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} \Lambda^p_{\text{sup}}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})$$

for $p \geq 0$. Note $r(\{D_p : \ p = 0, 1, 2, \dots\}) = \text{Kr.}\dim S^{\bullet}(\mathfrak{g}_{\bar{1}})$. The result follows by tensoring this resolution by M.

Theorem

Let $M \in \mathcal{F}$ and let $P_\bullet \twoheadrightarrow M$ be a minimal projective resolution. Then

$$c_{\mathcal{F}}(M) := r(P_\bullet) = r\left(\mathrm{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, \bigoplus S^{\dim P(S)})\right)$$

where the sum is over all simple modules S in $\mathcal{F}$, and $P(S)$ is the projective cover of S.

Type I Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (compatible $\mathbb{Z}$-grading) be a Type I classical Lie superalgebra. The Lie superalgebras $\mathfrak{gl}(m|n)$ and the simple Lie superalgebras of types $A(m,n)$ and $C(n)$ are Type I.

Let

$$\mathfrak{p}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \qquad \text{and} \qquad \mathfrak{p}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}.$$

Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_0$ and Borel subalgebra $\mathfrak{b}_0 \subseteq \mathfrak{g}_0$ so that $\mathfrak{h} \subseteq \mathfrak{b}_0$.

Kac and dual Kac modules

Let X^+ denote the parameterizing set of highest weights for the simple finite dimensional $\mathfrak{g}_0$-modules with respect to the pair $(\mathfrak{h}, \mathfrak{b}_0)$. For $\lambda \in X^+$, let $L_0(\lambda)$ denote the simple $\mathfrak{g}_0$-module of highest weight λ. View $L_0(\lambda)$ as a simple $\mathfrak{p}^{\pm}$–supermodule via inflation. Set

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} L_0(\lambda) \qquad \text{and} \qquad K^-(\lambda) = \mathrm{Hom}_{U(\mathfrak{p}^-)}(U(\mathfrak{g}), L_0(\lambda))$$

be the *Kac supermodule* and the *dual Kac supermodule*, respectively

The category $\mathcal{F}$ for Type I Lie superalgebras can be regarded as a highest weight category (as defined in Cline-Parshall-Scott using the Kac and dual Kac modules). This category is also self-injective.

Support varieties for $\mathfrak{g}_{\pm 1}$

Observe that $\mathfrak{g}_{\pm 1}$ is an abelian Lie superalgebra, thus

$$R = \mathrm{H}^\bullet(\mathfrak{g}_{\pm 1}, \mathbb{C}) = \mathrm{H}^\bullet(\mathfrak{g}_{\pm 1}, \{0\}; \mathbb{C}) \cong S(\mathfrak{g}_{\pm 1}^*)$$

Let $\mathcal{C}$ be the category of finite dimensional $\mathfrak{g}_{\pm 1}$-supermodules. If $M \in \mathcal{C}$, then one can define the $\mathfrak{g}_{\pm 1}$ *support variety* of M. Set

$$I_M = \{r \in R \mid r.m = 0 \text{ for all } m \in \mathrm{Ext}^\bullet_{\mathcal{F}}(M, M)\}$$

and then the support variety of M is

$$\begin{aligned} \mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) &= \mathrm{MaxSpec}\,(R/I_M) \\ &\cong \{x \in \mathfrak{g}_{\pm 1} \mid M \text{ is projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}. \end{aligned}$$

Here $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ is canonically isomorphic to the "rank variety", and $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ detects $\mathfrak{g}_{\pm 1}$ projectivity.

Criteria for Kac and dual Kac filtrations

Theorem

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra, and M be a $\mathfrak{g}$-supermodule. Then the following are equivalent.

(a) *M has a Kac filtration.*

(b) $\mathrm{Ext}^1_{\mathcal{F}}(M, K^-(\mu)) = 0$ *for all* $\mu \in X^+(T)$.

(c) $\mathrm{Ext}^1_{\mathfrak{g}_{-1}}(\mathbb{C}, M) = 0$.

(d) $\mathcal{V}_{\mathfrak{g}_{-1}}(M) = 0$.

Criteria for Kac and dual Kac filtrations

Theorem

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra, and M be a $\mathfrak{g}$-supermodule. Then the following are equivalent.

(a) *M has a dual Kac filtration.*

(b) $\mathrm{Ext}^1_{\mathcal{F}}(K^+(\mu), M) = 0$ *for all* $\mu \in X^+(T)$.

(c) $\mathrm{Ext}^1_{\mathfrak{g}_{+1}}(\mathbb{C}, M) = 0$.

(d) $\mathcal{V}_{\mathfrak{g}_{+1}}(M) = \{0\}$.

Definition

A module in $\mathcal{F}$ is *tilting* if and only if it has both a Kac and a dual Kac filtration.

Criteria for Projectivity

Theorem

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra and $M \in \mathcal{F}$. Then

(a) *M is projective if and only if $\mathcal{V}_{\mathfrak{g}_1}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}(M) = \{0\}$.*

(b) *Suppose that $\mathfrak{g}$ admits a strong duality τ. If $M^{\tau} \cong M$ then M is projective if and only if $\mathcal{V}_{\mathfrak{g}_1}(M) = \{0\}$.*

Corollary

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra. A module $M \in \mathcal{F}$ is projective if and only if M is tilting.

Connections with Duflo-Serganova Associated Varieties

Consider the subvariety of $\mathfrak{g}_{\bar{1}}$ given by

$$\mathcal{X} = \left\{ x \in \mathfrak{g}_{\bar{1}} \mid x^2 = [x, x]/2 = 0 \right\}.$$

For $M \in \mathcal{F}$, Duflo and Serganova defined the *associated variety*

$$\begin{aligned} \mathcal{X}_M &= \{x \in \mathcal{X} \mid \operatorname{Ker}(x)/\operatorname{Im}(x) \neq 0\} \\ &= \{x \in \mathcal{X} \mid M \text{ is projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}. \end{aligned}$$

Here x is considered as an operator from $M \to M$ by $x(m) = x.m$ with $x^2 = 0$.

Note that

$$\mathcal{V}_{\mathfrak{g}_{-1}}(M) \cup \mathcal{V}_{\mathfrak{g}_1}(M) \subseteq \mathcal{X}_M \tag{1}$$

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \mathcal{X}_M \cap \mathfrak{g}_{\pm 1}. \tag{2}$$

We can now recover a theorem of Duflo-Serganova:

Theorem

Let $\mathfrak{g}$ be a classical Type I Lie superalgebra with M in $\mathcal{F}$. Then $\mathcal{X}_M = 0$ if and only if M is projective.

Refinements of the Projectivity Test

Theorem

Let $\mathfrak{g}$ be a Type I classical Lie superalgebra and $M \in \mathcal{F}$. Let $\{x_i \mid i \in I\}$ be a set of orbit representatives for the minimal orbits[a] of the action of $G_{\bar{0}}$ on $\mathfrak{g}_1$ and $\{y_j \mid j \in J\}$ be a set of orbit representatives for the minimal orbits of the action of $G_{\bar{0}}$ on $\mathfrak{g}_{-1}$.

(a) *Then M is projective in $\mathcal{F}$ if and only if M is projective on restriction to $U(\langle x_i \rangle)$ for all $i \in I$ and to $U(\langle y_j \rangle)$ for all $j \in J$.*

(b) *Furthermore, assume that $\mathfrak{g}$ admits a strong duality τ and $M \cong M^{\tau}$. Then M is a projective in $\mathcal{F}$ if and only if M is projective on restriction to $U(\langle x_i \rangle)$ for all $i \in I$.*

[a] By minimal orbit, we mean minimal non-zero orbit with respect to the partial order on orbits given by containment in closures.

Further Questions

(1) If $P(S)$ is a projective cover for a simple module S in $\mathcal{F}$ then $P(S) \cong I(T)$ where $I(T)$ is the injective hull of a simple module T in $\mathcal{F}$. Can one find a general formula (like the one for finite-dimensional Hopf algebras) which describes the relationship between S and T?

(2) Given a simple module S in $\mathcal{F}$, can one give a concrete realization of $\text{End}_{\mathcal{F}}(P(S))$. An answer to this question may contribute to a solution of (1).

(3) One fundamental and elusive question is whether one can construct a “support variety” for a given module M in $\mathcal{F}$ whose dimension is equal to the complexity of M.