Quadratic unipotent blocks of general linear, unitary and symplectic groups

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$\mathbf{G}$ is a connected reductive algebraic group defined over $\mathbf{F}_q$,

$F : \mathbf{G} \to \mathbf{G}$ a Frobenius morphism,

$G = \mathbf{G}^F$ is a finite reductive group.

Examples: $GL(n, q)$, $U(n, q)$, $Sp(2n, q)$, $SO^{\pm}(2n, q)$

G has subgroups maximal tori, Levi subgroups (centralizers of tori)

Let ℓ be a prime not dividing q.

Suppose $\mathbf{L}$ is an F-stable Levi subgroup.

- The Deligne-Lusztig linear operator:

$$R_{\mathbf{L}}^{\mathbf{G}} : K_0(\overline{\mathbf{Q}}_l L) \to K_0(\overline{\mathbf{Q}}_l G).$$

- The unipotent characters of G are the irreducible characters χ in $R_{\mathbf{T}}^{\mathbf{G}}(1)$ as $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$.

If $\mathbf{L}$ is in an F-stable parabolic subgroup $\mathbf{P}$,

$R_{\mathbf{L}}^{\mathbf{G}}$ is just Harish-Chandra induction.

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Lusztig classification of complex characters is in good shape.

$\mathrm{Irr}(G) = \cup\, \mathcal{E}(G,(s))$, union of Lusztig series, $(s) \subset G^*$, a semisimple conjugacy class.

K a sufficiently large field of characteristic 0

$\mathcal{O}$ a complete discrete valuation ring with quotient field K

The ordinary characters or KG-modules are partitioned into blocks corresponding to the decomposition of $\mathcal{O}G$ into indecomposable two-sided ideals called block algebras.

G is a finite reductive group, e.g. a classical group. ℓ a prime not dividing q.

Problem: Describe the ℓ-blocks of G.

A unipotent block is a block which contains unipotent characters. Describe the unipotent blocks.

Let $G = GL(n, q)$, e the order of q mod ℓ. The unipotent characters of G are constituents of the permutation representation on the cosets of the subgroup B of upper triangular matrices. They are indexed by partitions of n. Say χ_λ corresponds to the partition λ.

Theorem (Fong-Srinivasan, 1982) χ_λ, χ_μ are in the same ℓ-block if and only if λ, μ have the same e-core.

Proof involves Deligne-Lusztig theory and Brauer theory. These two theories are compatible!

$G = Sp(2n, q), SO(2n+1, q), SO^{\pm}(2n, q)$,

A symbol Λ is a pair (S, T) of subsets of $\mathbf{N}$.

Notion of e-hooks, e-cohooks, e-cores of symbols defined.

$$\begin{pmatrix} 0 & & 1 & & 2 \\ & 1 & & 3 & \end{pmatrix}, \begin{pmatrix} 0 & & 1 & & 4 \\ & 1 & & 3 & \end{pmatrix}, \begin{pmatrix} & 0 & & 1 & \\ 1 & & 3 & & 4 \end{pmatrix}$$

The second symbol comes from the first by adding a 2-hook.

The third symbol comes form the first by adding 2-cohook.

In $G = CSp(2n, q), SO(2n+1, q), CSO^{\pm}(2n, q)$, unipotent characters are parameterized by symbols.

q and ℓ odd, e the order of q mod ℓ.

Unipotent blocks are again classified by e-cores of symbols. (Fong-Srinivasan,1989)

THEOREM ψ_{Λ_1}, ψ_{Λ_2} are in the same ℓ-block if and only if the symbols Λ_1, Λ_2 have the same e-core.

e-Harish-Chandra theory for unipotent characters: The Lusztig series $\mathcal{E}(G, 1)$ is partitioned into families.

The characters in a family are constituents of $R_L^G(\psi)$ where L is an "e-split Levi subgroup", ψ a unipotent "e-cuspidal" character of L.

Then (L, ψ) is called an e-cuspidal pair.

$e = 1$ gives the usual Harish-Chandra theory.

THEOREM (Cabanes-Enguehard) Let B be a unipotent block of G, ℓ odd. Then the unipotent characters in B are precisely the constituents of $R_L^G(\psi)$ where the pair (L, ψ) is e-cuspidal.

Thus we have a fit of Brauer theory and Lusztig theory. The subgroup $N_G(L)$ here plays the role of a "local subgroup".

EXAMPLE. GL(n,q): $L \cong T_1 \times T_2 \times \ldots T_r \times GL(m,q)$, where the T_i are tori of order $q^e - 1$ and $\psi = 1 \times \chi_\lambda$, λ an e-core.

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Arbitrary ℓ-block B of G determines a conjugacy class (s) in a dual group G^* of G, where $s \in G^*$ is an ℓ'-semi simple element. Then one hopes for a Jordan decomposition of blocks, i.e. a unipotent block of $C_{G^*}(s)$ sharing some properties with B.

Some modern problems of modular representation theory:

G is a finite reductive group, H some related group, e.g. another finite reductive group, $N_G(L)$, L Levi in G, or $C_{G^*}(s)$ for some s.
Block B of G, block b of H

- (Broué) Establish a perfect isometry between B and b (over K)
- (BADC) (Broué's abelian defect group conjecture) derived equivalence of blocks between $\mathcal{O}B$ and $\mathcal{O}b$
- Morita equivalence between $\mathcal{O}B$ and $\mathcal{O}b$

Bonnafé-Rouquier: If B corresponds to $s \in G^*$ where $C_{G^*}(s)$ is contained in a Levi subgroup, there is a Morita equivalence between B and a unipotent block of b.

Block B of G, block b of H :

A perfect isometry is a bijection between $K_0(B)$ and $K_0(b)$ preserving certain invariants of B and b.

Leads to:

- B and b have the same number of ordinary and modular irreducible characters
- Cartan matrices of the blocks B and b define the same integral quadratic form.

Some results on perfect isometries, when the defect group of the blocks are abelian:

- Broué, Malle, Michel: perfect isometries between unipotent blocks of finite reductive groups and normalizers of Levi subgroups (abelian defect groups)
- Rouquier: Between two symmetric groups ("equal weight")
- Enguehard: Between two general linear groups ("equal weight")

Stronger results due to Chuang-Rouquier: BADC for general linear groups

Question: Perfect isometries between groups of different Lie types? A possibility?

Example: $GL(4, q)$ and $Sp(4, q)$, ℓ divides $q + 1$. There is one block correspondence between principal blocks. But $GL(4, q)$ has 5 unipotent characters in one block, $Sp(4, q)$ has 6 unipotent characters, 5 in one block and 1 in one block.

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Inspiration

p-adic groups, James Arthur: "We shall describe a classification of automorphic representations of classical groups in terms of those of general linear groups (endoscopic group)"

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Enlarge the set of unipotent characters.

G is a finite reductive group.

$\mathrm{Irr}(G) = \cup\ \mathcal{E}(G,(s))$, union of Lusztig series, $(s) \subset G^*$, a semisimple conjugacy class.

(Waldspurger) If $s^2 = 1$, characters in $\mathcal{E}(G,(s))$ are called quadratic unipotent (special case: unipotent, $s = 1$).

$G_n = GL(n,q)$ or $U(n,q)$, q odd.

Quadratic unipotent characters are parameterized by pairs of partitions (μ_1, μ_2) of k_i, $i = 1, 2$ resp., with $k_1 + k_2 = n$.

$H_n = Sp(2n, q)$. Unipotent characters parameterized by (equivalence classes of) symbols.

Quadratic unipotent characters parameterized by (equivalence classes of) pairs of symbols (Λ_1, Λ_2) where

Λ_1: unordered symbol of rank k_1

Λ_2: ordered symbol of rank k_2, $k_1 + k_2 = n$.

$C_{G^*}(s)$ can be disconnected, e.g $(SO(2k_1+1) \times SO^{\pm}(2k_2)) \rtimes Z_2$

Notation: $\mathrm{Irr}(G_n)_{qu}$, $\mathrm{Irr}(H_n)_{qu}$ for quadratic unipotent characters, W_n is the Weyl group of type B_n.

Waldspurger's Parametrization of $\mathrm{Irr}(G_n)_{qu}$:

$$(\mu_1, \mu_2) \longleftrightarrow \{(m_1, m_2, \rho_1, \rho_2)\}$$

$$m_1, m_2 \in \mathbf{N}, \rho_i \in \mathrm{Irr}(W_{N_i}), i = 1, 2$$

$$m_1(m_1 + 1)/2 + m_2(m_2 + 1)/2 + 2N_1 + 2N_2 = n$$

Here m_i, ρ_i come from the 2-core and the 2-quotient of μ_i.

Example:

$\chi_{(21,1)}$ is 2-cuspidal (no 2-core), in $\mathrm{Irr}(GL(4))_{qu}$. In $GL(6)$, $\chi_{(41,1)}$ is obtained by Lusztig induction from $L = GL(4) \times T_{q^2-1}$, with $\rho_1 = (1, -)$. Then $(m_1, m_2, \rho_1, \rho_2) = (2, 1, (1, -), -)$.

$$\begin{pmatrix} * & * & * & * \\ * & + & + & \end{pmatrix} \to \begin{pmatrix} * & * & + & + \\ * & & & \end{pmatrix} \to \begin{pmatrix} * & * \\ * & \end{pmatrix}$$

Waldspurger's Parametrization of $\mathrm{Irr}(H_n)_{qu}$:

$$\mathrm{Irr}(H_n)_{qu} \longleftrightarrow \{(h_1, h_2, \rho_1, \rho_2)\}$$

$$h_1 \in \mathbf{N}, h_2 \in Z, \rho_i \in \mathrm{Irr}(W_{N_i}), i = 1, 2$$

$$h_1(h_1 + 1) + {h_2}^2 + N_1 + N_2 = n$$

Waldspurger's bijection:

$(m_1, m_2) \longleftrightarrow (h_1, h_2)$, where

$$m_1 = sup(h_1 + h_2, h_1 - h_2 - 1), m_2 = sup(h_1 - h_2, h_2 - h_1 - 1)$$

.

$$\{2 - \text{cuspidals} \in \mathrm{Irr}(G_n)_{qu}\} \longleftrightarrow \{1 - \text{cuspidals} \in \mathrm{Irr}(H_n)_{qu}\}$$

Extend bijection to

$$\{\mathrm{Irr}(G_n)_{qu}\} \longleftrightarrow \{\mathrm{Irr}(H_m)_{qu}\}$$

by

$$\{(m_1, m_2, \rho_1, \rho_2)\} \longleftrightarrow \{(h_1, h_2, \rho_1, \rho_2)\}$$

$$m_1(m_1+1)/2 + m_2(m_2+1)/2 + 2N_1 + 2N_2 = n$$

,

$$h_1(h_1+1) + h_2^2 + N_1 + N_2 = m$$

.

Example: $|\mathrm{Irr}(Sp(4,q))_{qu}| = 23$, bijection of 14 with $GL(4,q)$, 8 with $GL(3,q)$, 1 with $GL(2,q)$.

$\theta_{10} \in \mathrm{Irr}(Sp(4,q)) \longleftrightarrow \chi_{(1,1)} \in \mathrm{Irr}(GL(2,q))_{qu}$

Note θ_{10} unipotent, $\chi_{(1,1)} \in \mathcal{E}(G,(s))$ with s of order 2, $m_1 = 1, m_2 = 1, h_1 = 1, h_2 = 0$.

Example: $|\mathrm{Irr}(GL(4,q))_{qu}| = 20$, bijection of 14 with $Sp(4,q)$, 4 with $Sp(6,q)$, 2 with $Sp(8,q)$.

Two with $Sp(8,q)$ are $\chi_{(21,1)}$, $\chi_{(1,21)}$, 2-cuspidal, also correspond to cuspidal unipotent characters of $O^-(8,q)$. Here $m_1 = 2, m_2 = 1, h_1 = 0, h_2 = \pm 2$.

K a sufficiently large field of characteristic 0.

L_n the category of quadratic unipotent representations of G_n over K, M_n the same for H_n.

THEOREM With the usual inner product, there is an isometry between $\oplus_{n\geqslant 0}K_0(L_n)$ and $\oplus_{n\geqslant 0}K_0(M_n)$.

Also: Both isomorphic to $Z[N \times N] \times \oplus_{n,m\geqslant 0}K_0(\mathcal{H}_n - \mathrm{mod}) \times K_0(\mathcal{H}_m - \mathrm{mod})$, $\mathcal{H}_n$ Hecke algebra of type B_n.

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Recent work: J.Algebra 184 (1996) and 319 (2008).

Theorem on unipotent blocks of $G = Sp(2n, q)$, $SO(2n+1, q)$, $SO^{\pm}(2n, q)$ generalized to "quadratic unipotent" blocks.

EXAMPLE. $H_n = Sp(2n, q)$: quadratic unipotent characters in a block are constituents of $R_L^{H_n}(\psi)$,

$L \cong T_1 \times T_2 \times \dots T_{M_1} \times T_1 \times T_2 \times \dots T_{M_2} \times Sp(2m, q)$, where the T_i are tori of order $q^f - 1$ and $\psi = 1 \times \mathcal{E} \times \chi_{\Lambda_1, \Lambda_2}$, Λ_1 and Λ_2 are f-cores.

Quadratic unipotent blocks classified by e-cores of pairs of symbols and weights.

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Quadratic unipotent blocks classified by e-cores of pairs of symbols and weights.

Fix an odd prime ℓ, e the order of q mod ℓ, $e = 2f$.

Let f be odd. COMPARE:

$G_n = U(n, q)$ and $H_n = Sp(2n, q)$, $q > n$, ℓ divides $q^f - 1$

$G_n = GL(n, q)$, and $H_n = Sp(2n, q)$, $q > n$,
ℓ divides $q^f + 1$.

Also: $e = 2f$ where f is even, i.e. $e \equiv 0 \pmod 4$ and ℓ divides $q^f + 1$. Exclude $e \equiv 2 \pmod 4$.

THEOREM Let q, ℓ be odd, and $q > n$. There are ℓ-block correspondences between blocks B of G_n and blocks b of H_n as follows:

(i)ℓ divides $q^f - 1$, f odd, B a quadratic-unipotent ℓ-block of $U(n, q)$ and b a quadratic-unipotent ℓ-block of $Sp(2m, q)$,some m

(ii)ℓ divides $q^f + 1$, f odd, B a quadratic-unipotent ℓ-block of $GL(n, q)$ and b a quadratic-unipotent ℓ-block of $Sp(2m, q)$, some m

There is a natural bijection between quadratic-unipotent characters in B and b.

When the defect groups are abelian, the defect groups are isomorphic and there is a perfect isometry between B and b

Let $Bl(G_n)_{qu}$ (resp. $Bl(H_n)_{qu}$ be the set of quadratic unipotent blocks of G_n (resp. H_n), ℓ divides $q^f - 1$ or $q^f + 1$ as above.

There is a bijection

$$\coprod_{n \geqslant 0} Bl(G_n)_{qu} \leftrightarrow \coprod_{n \geqslant 0} Bl(H_n)_{qu},$$

such that if $B \to b$, there is a natural bijection between quadratic-unipotent characters in B and b.

IDEA

Use the following correspondences:

$B \longleftrightarrow 2f - \text{core}(\lambda_1, \lambda_2) \longleftrightarrow \{(m_1, m_2, \rho_1, \rho_2)\} \longleftrightarrow$
$\{(h_1, h_2, \rho_1, \rho_2)\} \longleftrightarrow f - \text{core}(\Lambda_1, \Lambda_2) \longleftrightarrow b$

$m_1(m_1+1)/2 + m_2(m_2+1)/2 + 2N_1 + 2N_2 = n,$
$h_1(h_1+1) + h_2^2 + N_1 + N_2 = m.$

IDEA

Perfect Isometries "across types":

Use the paper of [BMM] to get an isotypy from B to a local subgroup of G_n of the form $N_{G_n}(L, \lambda)$, then to a local subgroup of H_n, then to b.

Endoscopic groups

Enguehard has defined for a finite reductive group G, $s \in G*$, a group $G(s)$ (can be called an endoscopy group).

Example: For $H_n = Sp(2n, q)$, s with $s^2 = 1$, $H_n(s) = Sp(2m, q) \times O(2n - 2m, q)$.

We also have correspondences between *unipotent* blocks of suitable $G_n(s)$ and $H_n(s)$.

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Here $Bl(G_n)_u$ denotes the set of unipotent blocks of G_n.

There is a bijection

$$\coprod_{n_1,n_2\geqslant 0} Bl(G_{n_1}\times G_{n_2})_u \leftrightarrow \coprod_{n_1,n_2\geqslant 0} Bl(Sp_{2n_1}\times O_{2n_2})_u,$$

such that if $B\to b$, there is a natural bijection between quadratic-unipotent characters in B and b.

SUMMARY

$\oplus_{n\geqslant 0} K_0(GL_n - \mathrm{mod})_{qu} \cong \oplus_{n\geqslant 0} K_0(Sp_{2n} - \mathrm{mod})_{qu}$.

$\oplus_{n_1,n_2\geqslant 0} K_0((GL_{n_1} \times GL_{n_2}) - \mathrm{mod})_u \cong$
$\oplus_{n_1,n_2\geqslant 0} K_0((Sp_{2n_1} \times O_{2n_2}) - \mathrm{mod})_u$.

SUMMARY

For suitable ℓ: $\coprod_{n\geqslant 0} Bl(GL_n)_{qu} \leftrightarrow \coprod_{n\geqslant 0} Bl(Sp_{2n})_{qu}$

$\coprod_{n_1,n_2\geqslant 0} Bl(G_{n_1} \times G_{n_2})_u \leftrightarrow \coprod_{n_1,n_2\geqslant 0} Bl(Sp_{2n_1} \times O_{2n_2})_u$

What more can we say about this correspondence between blocks of general linear/unitary groups and blocks of symplectic groups?

Are corresponding blocks derived equivalent? Morita equivalent?

Has the symplectic group reached equal status with the general linear group, her "all-embracing majesty"?

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