Reduction of order for discrete systems

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Motivation

- A general approach to obtaining integrable hierarchies together with their underlying linear problems. It has provided hierarchies of partial and ordinary differential equations, lattices, differential-delay and discrete equations.

- We expected them to be Painlevé type equations but usually a reduction of order is needed. Use properties of completely integrable partial differential equations.

- Extension to the discrete case, i.e., use properties of integrable lattices to "reduce the order" of correponding discrete hierarchies.

(Joint work with A.Pickering)

Outline of the talk

- Introduction: the continuous case.
- Extension of the method to the discrete case.
  Identifying discrete Painlevé hierarchies.
- A Toda example.
- A Volterra example.
- Conclusions.

Introduction: the continuous case

The Korteweg-de Vries equation

$$v_t + (v_{zz} + 3v^2)_z = 0$$

The similarity reduction

$$v(z,t) = \frac{u(x)}{(3t)^{\frac{2}{3}}} \qquad x = \frac{z}{(3t)^{\frac{1}{3}}}$$

yields

$$(u_{xx} + 3u^2)_x - xu_x - 2u = 0$$

which integrates once to

$$(2u - x)u_{xx} + (2u - x)^2 u - (u_x)^2 + u_x = C \qquad \mathbf{P_{34}}$$

Both equations define the same function, but this function is better described as being defined by a second order equation.

We know that $\mathbf{P_{34}}$ is related to $\mathbf{P_{II}}$ **(second Painlevé equation)**

$$U_{xx} = 2U^3 + xU + \alpha$$

via the **Bäcklund transformation (BT)**

$$u = U_x - U^2 \qquad U = -\frac{u_x - \alpha}{2u - x} \qquad C = \alpha(1 - \alpha)$$

Think of the integration procedure in a different way

The first equation can also be written as

$$B[u](u - x/2) = 0$$

where

$$B[u] = \partial_x^3 + 4u\partial_x + 2u_x$$

is one of the Hamiltonian operators of the KdV hierarchy

This operator **factorizes under the Miura map**

$$u = F[U] = U_x - U^2$$

$$B[u] \mid_{u=F[U]} = F'[U]\tilde{B}(F'[U])^\dagger$$

$\tilde{B} = -\partial_x$ (the Hamiltonian operator of the mKdV hierarchy)

$F'[U] = \partial_x - 2U$ (the Fréchet derivative of $F[U]$)

$$F'[U]V = \left(\frac{\partial}{\partial\epsilon}F[U + \epsilon V]\right) \mid_{\epsilon=0}$$

a differential operator

$$F'[U] = \frac{\partial}{\partial U} + \frac{\partial}{\partial U_x}\partial_x + \frac{\partial}{\partial U_{xx}}\partial_x^2 + \dots$$

(Hamiltonian theory of PDEs)

Back to the ODE case.

The modified version of $B[u](u - x/2) = 0$ is

$$\tilde{B}(F'[U])^{\dagger}(U_x - U^2 - x/2) = 0$$

or

$$\partial_x(\partial_x + 2U)(U_x - U^2 - x/2) = 0$$

It integrates to give

$$(\partial_x + 2U)(U_x - U^2 - x/2) = \alpha - 1/2 \quad \text{or} \quad U_{xx} = 2U^3 + xU + \alpha$$

α is an arbitrary constant.

We have effected a reduction of order of the modified equation.

Question: How to relate this integrated modified equation (P_{II}) to the original equation?

Answer: Construct a **BT** between the integrated modified equation and an integrated version of the original equation.

$$\begin{aligned} U(2u - x) + (u_x - \alpha) &= 0 \\ u - U_x + U^2 &= 0 \end{aligned}$$

$$(2u - x)u_{xx} + (2u - x)^2 u - (u_x)^2 + u_x = \alpha(1 - \alpha)$$

$$
\begin{tikzcd}
B[u](u - x/2) = 0 \arrow[r, "\text{modification}"] \arrow[d, "\text{integration(unmod)}"'] & \tilde{B}(F'[U])^{\dagger}(U_x - U^2 - x/2) = 0 \arrow[d, "\text{integration(mod)}"] \\
P_{XXXIV} & P_{II} \arrow[l, "\text{BT}"']
\end{tikzcd}
$$

Advantage: The integration of the modified system is much simpler.

Underlying idea: Use features typical of integrable PDEs to reduce the order of ODEs (integrable or non-integrable).

Result: Identification of new Painlevé hierarchies.

Reduction of order via modification

(the continuous case)

Consider equations of the form

$$B[\mathbf{u}]\mathbf{L}[\mathbf{u}] = 0$$

B is a Hamiltonian operator.

Let $\mathbf{u} = \mathbf{F}[\mathbf{U}]$, ($\mathbf{U} = (U_0, \ldots, U_{N-1})^T$), be a Miura transformation such that:

(i) $B[\mathbf{u}]\,|_{\mathbf{u}=\mathbf{F}[\mathbf{U}]} = \mathbf{F}'[\mathbf{U}]\tilde{B}(\mathbf{F}'[\mathbf{U}])^\dagger$, where $\tilde{B} = A\partial_x$ for some non-singular constant matrix A;

(ii) $(\mathbf{F}'[\mathbf{U}])^\dagger\mathbf{L}[\mathbf{u}] + \gamma = 0$, with $\gamma = (\gamma_0, \cdots \gamma_{N-1})^T$ constant, is a linear system in the modified variables $U_0, \cdots U_{N-1}$, for which it has the unique solution $\mathbf{U} = \mathbf{G}[\mathbf{u}, \gamma]$.

Then

$$\begin{aligned}&(\mathbf{F}'[\mathbf{U}])^\dagger\mathbf{L}[\mathbf{u}] + \gamma = 0\\ &\mathbf{u} = \mathbf{F}[\mathbf{U}]\end{aligned}$$

defines a BT between the two systems

$$\begin{aligned}&(\mathbf{F}'[\mathbf{U}])^\dagger\mathbf{L}[\mathbf{F}[\mathbf{U}]] + \gamma = 0\\ &\mathbf{u} - \mathbf{F}[\mathbf{G}[\mathbf{u}, \gamma]] = 0\end{aligned}$$

and $\mathbf{u} - \mathbf{F}[\mathbf{G}[\mathbf{u}, \gamma]] = 0$ represents **an integrated version of the original system** $B[\mathbf{u}]\mathbf{L}[\mathbf{u}] = 0$.

The discrete case

$$\mathbf{U}_t = \mathcal{R}^n \mathbf{U}_y + \sum (\text{flows in } 1+1) + \sum (\text{non} - \text{isospectral terms})$$

$\mathbf{U} = \mathbf{U}(n,t,y)$ and $\mathcal{R}$ recursion operator of lattice hierarchy

An example based on the Volterra lattice

$$u_{t_m}^{(n)} = \left(\mathcal{R}^{(\mathrm{n})}\right)^m u_y^{(n)} + \sum_{j=0}^{m-1} \alpha_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j K_1^{(n)} + \sum_{j=0}^{m} \beta_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j u^{(n)}$$

$$\mathcal{R}^{(\mathrm{n})} = u^{(n)} \left(1 + E^{-1}\right) \left(u^{(n)} - u^{(n+1)} E^2\right) (E-1)^{-1} \left(u^{(n)}\right)^{-1}$$

$$K_1^{(n)} = u^{(n)} \left(u^{(n-1)} - u^{(n+1)}\right)$$

Linear problem

$$E\phi^{(n)} = \begin{pmatrix} 1 & u^{(n)} \\ 1/\lambda & 0 \end{pmatrix} \phi^{(n)}, \qquad \phi_{t_m}^{(n)} = \lambda^m \phi_y^{(n)} + H_m^{(n)} \phi^{(n)},$$

where $\lambda_{t_m} = \lambda^m \lambda_y + \Sigma_{j=0}^m \lambda^{m+1-j} \beta_j$

Reductions to:

1. $1+1$ differential-difference (lattice) hierarchies
2. $1+1$ differential-delay hierarchies
3. ordinary differential-delay hierarchies
4. discrete Painlevé hierarchies

Discrete Painlevé hierarchies

$$\sum_{j=0}^{m-1} \alpha_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j K_1^{(n)} + \sum_{j=0}^{m} \beta_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j u^{(n)} = 0$$

Local case $\beta_k = 0$, $k = 0, 1, \ldots, m-2$

$$\frac{1}{u^{(n)}} \left(\sum_{j=0}^{m-1} \alpha_{m-j} \left(\mathcal{R}^{(\mathrm{n})}\right)^j K_1^{(n)} + \beta_{m-1} \mathcal{R}^{(\mathrm{n})} u^{(n)} + \beta_m u^{(n)} \right) = 0$$

Example: $m = 2$

$$(E-1)\left[\alpha_1 u^{(n-1)} \left(u^{(n)} + u^{(n-1)} + u^{(n-2)}\right) - \alpha_2 u^{(n-1)} + \left(\frac{1}{2}\beta_2\right) n\right]$$
$$-\beta_1 \left[u^{(n-1)} + (E-1)\left((n-1)u^{(n-1)}\right)\right] - \omega_2 (-1)^n = 0$$

Continuum limit ($\omega_2 = 0$)

$$y''' + 6yy' + a_1 y' + 2g_0(xy' + 2y) + g_1 = 0$$

So discrete equation for $m = 2$ corresponds to similarity reduction of Korteweg-de Vries, but with additional parity-dependent term

Subcase (a) $\beta_1 = 0$

$$\alpha_1 u^{(n)} \left(u^{(n+1)} + u^{(n)} + u^{(n-1)}\right) - \alpha_2 u^{(n)}$$
$$+ \left(\frac{1}{2}\beta_2\right) n - \nu_2 - \mu_2 (-1)^n = 0$$

So includes dP_I as a special case

Subcase (b) $\beta_1 \neq 0$: can also be summed

$$u^{(n-1)}\Big[\alpha_1 u^{(n)} + \alpha_1 u^{(n-1)} - \beta_1 n + 2\gamma\Big]\Big[\alpha_1 u^{(n-1)} + \alpha_1 u^{(n-2)}$$
$$-\beta_1(n-1) + 2\gamma\Big] - \tilde{\beta}_2\Big[\alpha_1 u^{(n-1)} - C_2 - \frac{1}{2}\beta_1(n-1) + \gamma\Big] \times$$
$$\Big[\alpha_1 u^{(n-1)} + C_2 - \frac{1}{2}\beta_1 n + \gamma\Big] - \omega_2\Big[\alpha_1 u^{(n-1)}(-1)^{n-1}$$
$$+\gamma(-1)^{n-1} + B_2^{(n)}\Big] = 0$$

where $B_2^{(n)} = \begin{cases} \frac{1}{2}\beta_1 n, & n \text{ even}, \\ -\frac{1}{2}\beta_1(n-1), & n \text{ odd}. \end{cases}$

This is dP_{34} (generalization of known version)

Related by new BT to general dP_{II}

$$(1-(q^{(n)})^2)(q^{(n+1)}+q^{(n-1)})-(A_1+A_2 n)q^{(n)}-A_3-A_4(-1)^n = 0,$$

Example: $m = 3$

$$(E-1)\Big\{-\alpha_1 u^{(n-1)}\Big(u^{(n)}u^{(n+1)} + {u^{(n)}}^2 + 2u^{(n-1)}u^{(n)}$$
$$+u^{(n-2)}u^{(n-3)} + {u^{(n-2)}}^2 + 2u^{(n-1)}u^{(n-2)} + {u^{(n-1)}}^2 + u^{(n-2)}u^{(n)}\Big)$$
$$+\alpha_2 u^{(n-1)}(u^{(n)} + u^{(n-1)} + u^{(n-2)}) - \alpha_3 u^{(n-1)} + \left(\frac{1}{2}\beta_3\right) n\Big\}$$
$$-\beta_2\left[u^{(n-1)} + (E-1)\left((n-1)u^{(n-1)}\right)\right] - \omega_3(-1)^n = 0$$

- Special case $\beta_2 = 0$ sums to fourth order dP_I
- Case $\beta_2 \neq 0$ sums to fourth order dP_{34} (generalized version)
- This fourth order dP_{34} related by new BT to complete known fourth order dP_{II}

Identifying discrete Painlevé hierarchies

We have **extended our method of reduction of order to discrete equations**; a reduction of order can also be effected by using properties of integrable lattices.

We consider systems of N discrete equations in $\mathbf{p}_n = (p_{1,n}, \ldots, p_{N,n})^T$ of the form

$$\mathcal{B}[\mathbf{p}_n]\mathbf{K}[\mathbf{p}_n] = \mathbf{0}$$

$\mathcal{B}[\mathbf{p}_n]$ is a Hamiltonian operator and we have the Miura map $\mathbf{p}_n = \mathbf{M}[\mathbf{u}_n]$, ($\mathbf{u}_n = (u_{1,n}, \ldots, u_{N,n})^T$) for completely integrable lattice systems.

The differences with the continuous case:

(A) We allow the Hamiltonian operator of the modified system, $\tilde{\mathcal{B}}[\mathbf{u}_n]$, to depend on $\mathbf{u}_n$. In particular, in the factorization

$$\mathcal{B}[\mathbf{p}_n]\Big|_{\mathbf{p}_n=\mathbf{M}[\mathbf{u}_n]} = \mathbf{M}'[\mathbf{u}_n]\tilde{\mathcal{B}}[\mathbf{u}_n]\left(\mathbf{M}'[\mathbf{u}_n]\right)^{\dagger},$$

we assume that

$$\tilde{\mathcal{B}}[\mathbf{u}_n] = \mathcal{C}[\mathbf{u}_n]\mathcal{D}\mathcal{C}[\mathbf{u}_n],$$

and that it is through the constant-coefficient skew-adjoint matrix operator $\mathcal{D}$ — which here plays the role played by $A\frac{d}{dx}$ in the differential case — that $\tilde{\mathcal{B}}[\mathbf{u}_n]$ depends on E.

The summed modified system has then the form

$$\mathcal{C}[\mathbf{u}_n]\,(\mathbf{M}'[\mathbf{u}_n])^{\dagger}\,\mathbf{K}[\mathbf{M}[\mathbf{u}_n]] + \omega(n) = \mathbf{0}.$$

(B) When we replace $\mathbf{M}[\mathbf{u}_n]$ by $\mathbf{p}_n$, to form together with the Miura map the BT

$$\begin{aligned} \mathcal{C}[\mathbf{u}_n]\,(\mathbf{M}'[\mathbf{u}_n])^{\dagger}\,\mathbf{K}[\mathbf{p}_n] + \omega(n) &= \mathbf{0} \\ \mathbf{p}_n - \mathbf{M}[\mathbf{u}_n] &= \mathbf{0} \end{aligned}$$

we assume that the product $\mathcal{C}[\mathbf{u}_n]\,(\mathbf{M}'[\mathbf{u}_n])^{\dagger}$ is such that the first part of the BT is a linear algebraic system for the modified variables $\mathbf{u}_n$ (perhaps after using the Miura map) having the unique solution

$$\mathbf{u}_n = \mathbf{G}[\mathbf{p}_n, \omega(n)]$$

We obtain a BT between the summed modified system and the summation of the original (unmodified) system

$$\mathbf{p}_n - \mathbf{M}[\mathbf{G}[\mathbf{p}_n, \omega(n)]] = \mathbf{0}.$$

The reduction of order depends on $\mathcal{D}$.

A Toda example

We consider two-component systems in $\mathbf{p}_n = (p_n, q_n)^T$ of the form

$$\mathcal{B}[\mathbf{p}_n]\mathbf{K}[\mathbf{p}_n] = \mathbf{0}$$

It arises as a reduction to discrete equations of a non-isospectral **Toda lattice** hierarchy.

$\mathcal{B}[\mathbf{p}_n]$ is the second of the three Hamiltonian operators of the Toda lattice hierarchy (Kupershmidt),

$$\mathcal{B}[\mathbf{p}_n] = \begin{pmatrix} Eq_n - q_nE^{-1} & p_n(E-1)q_n \\ q_n(1-E^{-1})p_n & q_n(E-E^{-1})q_n \end{pmatrix} \qquad E^k f_n = f_{n+k}$$

Miura map

$$\mathbf{p}_n = \mathbf{M}[\mathbf{u}_n] = \begin{pmatrix} u_n + v_n \\ u_{n-1}v_n \end{pmatrix}, \qquad \mathbf{u}_n = (u_n, v_n)^T$$

Factorization

$$\mathcal{B}[\mathbf{p}_n]\Big|_{\mathbf{p}_n=\mathbf{M}[\mathbf{u}_n]} = \mathbf{M}'[\mathbf{u}_n]\tilde{\mathcal{B}}[\mathbf{u}_n]\,(\mathbf{M}'[\mathbf{u}_n])^\dagger$$

$\mathbf{M}'[\mathbf{u}_n]$ is the Fréchet derivative of the Miura map

$$\mathbf{M}'[\mathbf{u}_n] = \begin{pmatrix} 1 & 1 \\ v_nE^{-1} & u_{n-1} \end{pmatrix} \qquad (\mathbf{M}'[\mathbf{u}_n])^\dagger = \begin{pmatrix} 1 & Ev_n \\ 1 & u_{n-1} \end{pmatrix}$$

$\tilde{\mathcal{B}}[\mathbf{u}_n]$ Hamiltonian operator of the modified Toda lattice hierarchy

$$\tilde{\mathcal{B}}[\mathbf{u}_n] = \begin{pmatrix} 0 & u_n(E-1)v_n \\ v_n(1-E^{-1})u_n & 0 \end{pmatrix}$$

that can be written in the form

$$\tilde{\mathcal{B}}[\mathbf{u}_n] = \mathcal{C}[\mathbf{u}_n]\mathcal{D}\mathcal{C}[\mathbf{u}_n] = \begin{pmatrix} u_n & 0 \\ 0 & v_n \end{pmatrix} \begin{pmatrix} 0 & E-1 \\ 1-E^{-1} & 0 \end{pmatrix} \begin{pmatrix} u_n & 0 \\ 0 & v_n \end{pmatrix}.$$

THE RESULT

- **Summed modified system**

$$\begin{pmatrix} u_n & 0 \\ 0 & v_n \end{pmatrix} \begin{pmatrix} 1 & Ev_n \\ 1 & u_{n-1} \end{pmatrix} \begin{pmatrix} K_1[\mathbf{M}[\mathbf{u}_n]] \\ K_2[\mathbf{M}[\mathbf{u}_n]] \end{pmatrix} + \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\omega = (\omega_1, \omega_2)^T$ is an arbitrary constant and $\mathbf{K} = (K_1, K_2)^T$

- **The BT**

$$\begin{pmatrix} u_n & 0 \\ 0 & v_n \end{pmatrix} \begin{pmatrix} 1 & Ev_n \\ 1 & u_{n-1} \end{pmatrix} \begin{pmatrix} K_1[\mathbf{p}_n] \\ K_2[\mathbf{p}_n] \end{pmatrix} + \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} - \begin{pmatrix} u_n + v_n \\ u_{n-1}v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It can be rewritten as

$$
\begin{aligned}
u_n K_1[\mathbf{p}_n] + q_{n+1} E K_2[\mathbf{p}_n] + \omega_1 &= 0 \\
v_n K_1[\mathbf{p}_n] + q_n K_2[\mathbf{p}_n] + \omega_2 &= 0 \\
p_n - (u_n + v_n) &= 0 \\
q_n - u_{n-1} v_n &= 0
\end{aligned}
$$

Linear algebraic system for the modified variables $\mathbf{u}_n = (u_n, v_n)^T$ with unique solution

$$
\begin{aligned}
u_n &= -\frac{q_{n+1} E K_2[\mathbf{p}_n] + \omega_1}{K_1[\mathbf{p}_n]}, \\
v_n &= -\frac{q_n K_2[\mathbf{p}_n] + \omega_2}{K_1[\mathbf{p}_n]}
\end{aligned}
$$

Substituting u_n and v_n into the Miura map we obtain

- **Summed unmodified system**

$$
p_n K_1[\mathbf{p}_n] + (E+1) q_n K_2[\mathbf{p}_n] + (\omega_1 + \omega_2) = 0
$$

$$
\begin{aligned}
q_n \Big(K_1[\mathbf{p}_n] E^{-1} K_1[\mathbf{p}_n] + p_n K_1[\mathbf{p}_n] K_2[\mathbf{p}_n] + q_{n+1} K_2[\mathbf{p}_n] E K_2[\mathbf{p}_n] \Big) \\
- \omega_1 \omega_2 = 0
\end{aligned}
$$

We have constructed a BT between the summed modified system and the summation of our original system

Application: An asymmetric $\mathbf{dP_I}$ hierarchy

We obtained the hierarchy of discrete equations

$$\sum_{j=0}^{m} \gamma_{m-j-1} \mathcal{Q}_n^j \begin{pmatrix} q_{n+1} - q_n \\ q_n(p_n - p_{n-1}) \end{pmatrix} + \beta_{m-1} \begin{pmatrix} p_n \\ 2q_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(Gordoa, Pickering and Zhu, 07)

all γ_k and β_{m-1} are constant, β_{m-1} being assumed nonzero, and where $\mathcal{Q}_n$ is the recursion operator of the standard Toda lattice hierarchy

$$\mathcal{Q}_n = \mathcal{B}[\mathbf{p}_n]\mathcal{A}^{-1}[\mathbf{p}_n]$$

with $\mathcal{A}[\mathbf{p}_n]$ given by

$$\mathcal{A}[\mathbf{p}_n] = \begin{pmatrix} 0 & (1-E)q_n \\ q_n(E^{-1}-1) & 0 \end{pmatrix}$$

The hierarchy can also be written in the form

$$\mathcal{B}[\mathbf{p}_n] \begin{pmatrix} K_1[\mathbf{p}_n] \\ K_2[\mathbf{p}_n] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} K_1[\mathbf{p}_n] \\ K_2[\mathbf{p}_n] \end{pmatrix} = \sum_{j=0}^{m} \gamma_{m-j-1} L_{j+1,n} + \beta_{m-1} \begin{pmatrix} 0 \\ \frac{n-1}{q_n} \end{pmatrix},$$

Recursion relation

$$\mathcal{B}[\mathbf{p}_n] L_{j,n} = \mathcal{A}[\mathbf{p}_n] L_{j+1,n} \quad j \geq 1$$

$$L_{1,n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad L_{2,n} = \begin{pmatrix} -p_n \\ -1 \end{pmatrix}, \qquad L_{3,n} = \begin{pmatrix} q_{n+1} + q_n + p_n^2 \\ p_n + p_{n-1} \end{pmatrix}$$

Each member of the hierarchy has been summed twice

Example: $m = 1$

$$\begin{pmatrix} K_1[\mathbf{p}_n] \\ K_2[\mathbf{p}_n] \end{pmatrix} = \gamma_{-1} \begin{pmatrix} -p_n \\ -1 \end{pmatrix} + \gamma_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_0 \begin{pmatrix} 0 \\ \frac{n-1}{q_n} \end{pmatrix}$$

We obtain

$$\gamma_{-1}(q_{n+1} + q_n + p_n^2) - \gamma_0 p_n - \beta_0(2n-1) - a_1 - b_1 = 0$$

$$\begin{aligned} q_n(\gamma_{-1}p_n - \gamma_0)(\gamma_{-1}p_{n-1} - \gamma_0) - (\gamma_{-1}q_n - \beta_0(n-1))^2 \\ + (a_1 + b_1)(\gamma_{-1}q_n - \beta_0(n-1)) - a_1 b_1 = 0 \end{aligned}$$

Equivalent, under the BT

$$\begin{aligned} p_n &= u_n + v_n \\ q_n &= u_{n-1} v_n \\ u_n &= -\frac{\gamma_{-1}q_{n+1} - \beta_0 n - a_1}{\gamma_{-1}p_n - \gamma_0} \\ v_n &= -\frac{\gamma_{-1}q_n - \beta_0(n-1) - b_1}{\gamma_{-1}p_n - \gamma_0} \end{aligned}$$

to the system

$$\begin{aligned} \gamma_{-1}u_n(u_n + v_{n+1} + v_n) - \gamma_0 u_n - \beta_0 n - a_1 &= 0 \\ \gamma_{-1}v_n(u_n + u_{n-1} + v_n) - \gamma_0 v_n - \beta_0(n-1) - b_1 &= 0 \end{aligned}$$

the asymmetric dP_I equation

A Volterra example

We consider one-component systems in u_n of the form

$$\Big(\mathcal{B}[u_n] + c\mathcal{A}[u_n]\Big)K[u_n] = 0$$

(u_n are the unmodified variables and q_n the modified variables)

$\mathcal{B}[u_n]$ and $\mathcal{A}[u_n]$ are the Hamiltonian operators

$$\begin{aligned}\mathcal{A}[u_n] &= u_n(E^{-1} - E)u_n \\ \mathcal{B}[u_n] &= u_n(1+E)(u_nE - E^{-1}u_n)(1+E^{-1})u_n\end{aligned}$$

$c \neq 0$ constant

It is the combination $\mathcal{B}[u_n] + c\mathcal{A}[u_n]$ which plays the role of $\mathcal{B}[\mathbf{p}_n]$

Under the **Miura map**

$$u_n = M[q_n] = \frac{1}{4}c(1-q_n)(1+q_{n+1}),$$

we have the **factorization**

$$\Big(\mathcal{B}[u_n] + c\mathcal{A}[u_n]\Big)\Bigg|_{u_n=M[q_n]} = M'[q_n]\tilde{\mathcal{B}}[q_n]\,(M'[q_n])^\dagger$$

$$M'[q_n] = \frac{1}{4}c\Big((1-q_n)E - (1+q_{n+1})\Big)$$

$\tilde{\mathcal{B}}[q_n]$ is the Hamiltonian operator of the modified Volterra lattice hierarchy,

$$\tilde{\mathcal{B}}[q_n] = -\frac{1}{4}c(q_n^2 - 1)(E - E^{-1})(q_n^2 - 1).$$

The form of $\mathcal{D} = -\frac{1}{4}c(E - E^{-1})$ allows **a double summation**, with $\omega(n) = -\gamma_1 - \gamma_2(-1)^n$, ($\gamma_1$ and γ_2 two arbitrary constants).

THE RESULT

- **Summed modified system**

$$(q_n^2-1)\frac{1}{4}c\Big(E^{-1}(1-q_n)-(1+q_{n+1})\Big)K[M[q_n]]-\gamma_1-\gamma_2(-1)^n = 0.$$

- **The BT**

$$\begin{aligned}(q_n - 1)u_{n-1}E^{-1}K[u_n] + (q_n + 1)u_nK[u_n] - \gamma_1 - \gamma_2(-1)^n &= 0\\ u_n - \frac{1}{4}c(1 - q_n)(1 + q_{n+1}) &= 0\end{aligned}$$

A linear algebraic equation for q_n. Elimination of q_n gives

- **Summed unmodified system**

$$\begin{aligned}&u_n\Big(u_nK[u_n] + u_{n+1}EK[u_n]\Big)\Big(u_nK[u_n] + u_{n-1}E^{-1}K[u_n]\Big)\\ &-\frac{1}{4}c\Big(2u_nK[u_n] - \gamma_1 - \gamma_2(-1)^n\Big)\Big(2u_nK[u_n] + \gamma_1 - \gamma_2(-1)^n\Big) = 0\end{aligned}$$

Application: A generalized $\mathbf{dP_{34}}$ hierarchy

The hierarchy of integrable discrete equations

$$\sum_{j=0}^{m-1} \alpha_{m-j}\mathcal{R}_n^j\mathcal{K}_{1,n} + \beta_{m-1}\mathcal{R}_n u_n + \beta_m u_n = 0,$$

β_{m-1}, β_m and all α_k are constant

$$\mathcal{R}_n = \mathcal{B}[u_n]\mathcal{A}^{-1}[u_n] = u_n(1+E^{-1})(u_n - u_{n+1}E^2)(E-1)^{-1}u_n^{-1}$$

the recursion operator of the standard Volterra lattice hierarchy.

$$\mathcal{K}_{1,n} = [u_n(u_{n-1} - u_{n+1})]$$

$$\mathcal{R}_n u_n = u_n\Big((n-2)u_{n-1} - u_n - (n+1)u_{n+1}\Big).$$

Setting $c = \tilde{\beta}_m = \beta_m/\beta_{m-1}$ we write the hierarchy as

$$\Big(\mathcal{B}[u_n] + \tilde{\beta}_m\mathcal{A}[u_n]\Big)K[u_n] = 0,$$

where

$$K[u_n] = \sum_{j=0}^{m-1} \alpha_{m-j}G_{j+1,n} + \beta_{m-1}\left[\frac{1}{u_n}\left(\frac{1}{4} - \frac{n}{2}\right)\right]$$

and

$$G_{j+1,n} = \sum_{k=1}^{j+1}(-\tilde{\beta}_m)^{j+1-k}L_{k,n}.$$

$L_{j,n}$ satisfies the recurrence $\mathcal{B}[u_n]L_{j,n} = \mathcal{A}[u_n]L_{j+1,n}$, $j \geq 1$

$$L_{1,n} = -\frac{1}{2}\frac{1}{u_n}, \qquad L_{2,n} = 1, \qquad L_{3,n} = -(u_n + u_{n+1} + u_{n-1})$$

We set $\gamma_1 = a_m$, $\gamma_2 = b_m$, relabelled according to the position of the equation in the hierarchy.

Each member of the hierarchy has been summed twice

We obtain **a BT** between the summed modified hierarchy **(a generalized $\mathbf{dP_{II}}$ hierarchy which includes parity-dependent terms)** and our original hierarchy summed twice **(a generalized $\mathbf{dP_{34}}$ hierarchy)**.

We have constructed (and identify) two discrete Painlevé hierarchies together with the BT relating them.

The simplest nontrivial example: $m = 2$

We start point

$$K[u_n] = \alpha_1 G_{2,n} + \alpha_2 G_{1,n} + \beta_1 \left[\frac{1}{u_n}\left(\frac{1}{4} - \frac{n}{2}\right)\right]$$

with

$$G_{2,n} = L_{2,n} - \tilde{\beta}_2 L_{1,n} = 1 + \tilde{\beta}_2 \frac{1}{2}\frac{1}{u_n}, \qquad G_{1,n} = L_{1,n} = -\frac{1}{2}\frac{1}{u_n}$$

$$u_n K[u_n] = \alpha_1 u_n - \frac{1}{2}\beta_1 n + \gamma - \frac{1}{4}\beta_1,$$

and where

$$\gamma = \frac{1}{2}\left(-\alpha_2 + \beta_1 + \alpha_1 \tilde{\beta}_2\right).$$

The result for this example is the BT

$$
\begin{aligned}
u_n &= \frac{1}{4}\tilde{\beta}_2(1-q_n)(1+q_{n+1}) \\
q_n &= \frac{\alpha_1(u_{n-1}-u_n)+\frac{1}{2}\beta_1+a_2+b_2(-1)^n}{\alpha_1(u_{n-1}+u_n)+2\gamma-\beta_1 n}
\end{aligned}
$$

between the dP_{II} equation

$$
\frac{1}{4}\alpha_1\tilde{\beta}_2(1-q_n^2)(q_{n+1}+q_{n-1})+2\gamma q_n-\beta_1 n q_n-\frac{1}{2}\beta_1-a_2-b_2(-1)^n=0
$$

and the dP_{34} equation

$$
\begin{aligned}
u_n\Big(\alpha_1(u_{n+1}+u_n)-\beta_1(n+1)+2\gamma\Big)\Big(\alpha_1(u_n+u_{n-1})-\beta_1 n+2\gamma\Big) \\
-\frac{1}{4}\tilde{\beta}_2\Big[\Big(2\alpha_1 u_n-\beta_1 n+2\gamma-\frac{1}{2}\beta_1-b_2(-1)^n\Big)^2-a_2^2\Big] &= 0
\end{aligned}
$$

Conclusions

- A method of reduction of order for discrete systems having a certain structure. For both examples the reduction of order is by 2 and allows the identification of new Painlevé hierarchies.

- Characteristic features of integrable PDEs and lattices have implications for ODEs and discrete equations (e.g. Hamiltonian structures and BTs).

- What is important is the factorization of $B[\mathbf{u}]$ and not the particular form of $L[\mathbf{u}]$; non-integrable systems can be considered.

- Construct and identify new continuous and discrete Painlevé hierarchies.