Reduction of order for discrete systems

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Motivation

• A general approach to obtaining integrable hierarchies together with their underlying linear problems. It has provided hierarchies of partial and ordinary differential equations, lattices, differential-delay and discrete equations.

• We expected them to be Painlevé type equations but usually a reduction of order is needed. Use properties of completely integrable partial differential equations.

• Extension to the discrete case, i.e., use properties of integrable lattices to “reduce the order” of corresponding discrete hierarchies.

(Joint work with A. Pickering)
Outline of the talk

• Introduction: the continuous case.

• Extension of the method to the discrete case.
  Identifying discrete Painlevé hierarchies.

• A Toda example.

• A Volterra example.

• Conclusions.
Introduction: the continuous case

The Korteweg-de Vries equation

\[ v_t + (v_{zz} + 3v^2)_z = 0 \]

The similarity reduction

\[ v(z, t) = \frac{u(x)}{(3t)^\frac{2}{3}} \quad x = \frac{z}{(3t)^\frac{1}{3}} \]

yields

\[ (u_{xx} + 3u^2)_x - xu_x - 2u = 0 \]

which integrates once to

\[ (2u - x)u_{xx} + (2u - x)^2 u - (u_x)^2 + u_x = C \quad P_{34} \]

Both equations define the same function, but this function is better described as being defined by a second order equation.

We know that \( P_{34} \) is related to \( P_{II} \) (second Painlevé equation)

\[ U_{xx} = 2U^3 + xU + \alpha \]

via the Bäcklund transformation (BT)

\[ u = U_x - U^2 \quad U = -\frac{u_x - \alpha}{2u - x} \quad C = \alpha(1 - \alpha) \]
Think of the integration procedure in a different way

The first equation can also be written as

\[ B[u](u - x/2) = 0 \]

where

\[ B[u] = \partial_x^3 + 4u\partial_x + 2u_x \]

is one of the Hamiltonian operators of the KdV hierarchy.

This operator **factorizes** under the Miura map

\[ u = F[U] = U_x - U^2 \]

\[ B[u] \big|_{u=F[U]} = F'[U] \tilde{B}(F'[U])^\dagger \]

\( \tilde{B} = -\partial_x \) (the Hamiltonian operator of the mKdV hierarchy)

\[ F'[U] = \partial_x - 2U \) (the Fréchet derivative of \( F[U] \))

\[ F'[U]V = \left( \frac{\partial}{\partial \epsilon} F[U + \epsilon V] \right) \big|_{\epsilon=0} \]

a differential operator

\[ F'[U] = \frac{\partial}{\partial U} + \frac{\partial}{\partial U_x} \partial_x + \frac{\partial}{\partial U_{xx}} \partial_x^2 + \ldots \]

(Hamiltonian theory of PDEs)
Back to the ODE case.

The modified version of $B[u](u - x/2) = 0$ is

$$\tilde{B} (F'[U])^\dagger (U_x - U^2 - x/2) = 0$$

or

$$\partial_x (\partial_x + 2U)(U_x - U^2 - x/2) = 0$$

It integrates to give

$$(\partial_x + 2U)(U_x - U^2 - x/2) = \alpha - 1/2 \quad \text{or} \quad U_{xx} = 2U^3 + xU + \alpha$$

$\alpha$ is an arbitrary constant.

We have effected a reduction of order of the modified equation.

**Question:** How to relate this integrated modified equation ($P_{II}$) to the original equation?

**Answer:** Construct a BT between the integrated modified equation and an integrated version of the original equation.

$$U(2u - x) + (u_x - \alpha) = 0$$

$$u - U_x + U^2 = 0$$

$$(2u - x)u_{xx} + (2u - x)^2 u - (u_x)^2 + u_x = \alpha(1 - \alpha)$$
\[ B[u](u - x/2) = 0 \quad \xrightarrow{\text{modification}} \quad \tilde{B}(F'[U])\ddagger(U_x - U^2 - x/2) = 0 \]

**Advantage:** The integration of the modified system is much simpler.

**Underlying idea:** Use features typical of integrable PDEs to reduce the order of ODEs (integrable or non-integrable).

**Result:** Identification of new Painlevé hierarchies.
Reduction of order via modification

(the continuous case)

Consider equations of the form

\[ B[u]L[u] = 0 \]

\( B \) is a Hamiltonian operator.

Let \( u = F[U] \), \( U = (U_0, \ldots, U_{N-1})^T \), be a Miura transformation such that:

(i) \( B[u] \big|_{u=F[U]} = F'[U] \tilde{B}(F'[U])^\dagger \), where \( \tilde{B} = A \partial_x \) for some non-singular constant matrix \( A \);

(ii) \( (F'[U])^\dagger L[u] + \gamma = 0 \), with \( \gamma = (\gamma_0, \cdots \gamma_{N-1})^T \) constant, is a linear system in the modified variables \( U_0, \cdots U_{N-1} \), for which it has the unique solution \( U = G[u, \gamma] \).

Then

\[ (F'[U])^\dagger L[u] + \gamma = 0 \]
\[ u = F[U] \]

defines a BT between the two systems

\[ (F'[U])^\dagger L[F[U]] + \gamma = 0 \]
\[ u - F[G[u, \gamma]] = 0 \]

and \( u - F[G[u, \gamma]] = 0 \) represents an integrated version of the original system \( B[u]L[u] = 0 \).
The discrete case

\( U_t = \mathcal{R}^n U_y + \sum (\text{flows in } 1 + 1) + \sum (\text{non-isospectral terms}) \)

\( U = U(n, t, y) \) and \( \mathcal{R} \) recursion operator of lattice hierarchy

An example based on the Volterra lattice

\[
\begin{align*}
\displaystyle u^{(n)}_{m} &= (\mathcal{R}^{(n)})^m u^{(n)}_y + \sum_{j=0}^{m-1} \alpha_{m-j} (\mathcal{R}^{(n)})^j K_1^{(n)} + \sum_{j=0}^{m} \beta_{m-j} (\mathcal{R}^{(n)})^j u^{(n)} \\
\mathcal{R}^{(n)} &= u^{(n)} (1 + E^{-1}) (u^{(n)} - u^{(n+1)} E^2) (E - 1)^{-1} (u^{(n)})^{-1} \\
K_1^{(n)} &= u^{(n)} (u^{(n-1)} - u^{(n+1)})
\end{align*}
\]

Linear problem

\[
E \phi^{(n)} = \begin{pmatrix} 1 & u^{(n)} \\ 1/\lambda & 0 \end{pmatrix} \phi^{(n)} , \quad \phi_{t_m}^{(n)} = \lambda^m \phi_y^{(n)} + H_m^{(n)} \phi^{(n)},
\]

where \( \lambda_{t_m} = \lambda^m \lambda_y + \sum_{j=0}^{m} \lambda^{m+1-j} \beta_j \)

Reductions to:

1. 1 + 1 differential-difference (lattice) hierarchies
2. 1 + 1 differential-delay hierarchies
3. ordinary differential-delay hierarchies
4. discrete Painlevé hierarchies
Discrete Painlevé hierarchies

\[ \sum_{j=0}^{m-1} \alpha_{m-j} (\mathcal{R}(n))^j K_1^{(n)} + \sum_{j=0}^{m} \beta_{m-j} (\mathcal{R}(n))^j u^{(n)} = 0 \]

Local case \( \beta_k = 0, \ k = 0, 1, \ldots, m - 2 \)

\[ \frac{1}{u^{(n)}} \left( \sum_{j=0}^{m-1} \alpha_{m-j} (\mathcal{R}(n))^j K_1^{(n)} + \beta_{m-1} \mathcal{R}(n) u^{(n)} + \beta_m u^{(n)} \right) = 0 \]

Example: \( m = 2 \)

\[ (E - 1) \left[ \alpha_1 u^{(n-1)} (u^{(n)} + u^{(n-1)} + u^{(n-2)}) - \alpha_2 u^{(n-1)} + \left( \frac{1}{2} \beta_2 \right) n \right] \\ - \beta_1 [u^{(n-1)} + (E - 1) ((n - 1) u^{(n-1)})] - \omega_2 (-1)^n = 0 \]

Continuum limit \( (\omega_2 = 0) \)

\[ y''' + 6yy' + a_1 y' + 2g_0(xy' + 2y) + g_1 = 0 \]

So discrete equation for \( m = 2 \) corresponds to similarity reduction of Korteweg-de Vries, but with additional parity-dependent term

Subcase (a) \( \beta_1 = 0 \)

\[ \alpha_1 u^{(n)} (u^{(n+1)} + u^{(n)} + u^{(n-1)}) - \alpha_2 u^{(n)} \\ + \left( \frac{1}{2} \beta_2 \right) n - \nu_2 - \mu_2 (-1)^n = 0 \]

So includes dP_I as a special case
Subcase (b) \( \beta_1 \neq 0 \): can also be summed

\[
\begin{align*}
u^{(n-1)} &\left[ \alpha_1 u^{(n)} + \alpha_1 u^{(n-1)} - \beta_1 n + 2\gamma \right] \left[ \alpha_1 u^{(n-1)} + \alpha_1 u^{(n-2)} \right. \\
- \beta_1 (n-1) + 2\gamma &- \bar{\beta}_2 \left[ \alpha_1 u^{(n-1)} - C_2 - \frac{1}{2} \beta_1 (n-1) + \gamma \right] \times \\
\left. \left[ \alpha_1 u^{(n-1)} + C_2 - \frac{1}{2} \beta_1 n + \gamma \right] - \omega_2 \left[ \alpha_1 u^{(n-1)}(-1)^{n-1} \right. \\
+ \gamma (-1)^{n-1} + B_2^{(n)} \right] = 0
\end{align*}
\]

where

\[
B_2^{(n)} = \begin{cases} \\
\frac{1}{2} \beta_1 n, & n \text{ even}, \\
-\frac{1}{2} \beta_1 (n-1), & n \text{ odd}.
\end{cases}
\]

This is \( dP_{34} \) (generalization of known version)

Related by new BT to general \( dP_{II} \)

\[
(1 - (q^{(n)})^2)(q^{(n+1)} + q^{(n-1)}) - (A_1 + A_2 n) q^{(n)} - A_3 - A_4 (-1)^n = 0,
\]

Example: \( m = 3 \)

\[
(E - 1) \left\{ - \alpha_1 u^{(n-1)} \left( u^{(n)} u^{(n+1)} + u^{(n)} + 2u^{(n-1)} u^{(n)} \right. \\
+ u^{(n-2)} u^{(n-3)} + u^{(n-2)} + 2u^{(n-1)} u^{(n-2)} + u^{(n-1)} + u^{(n-2)} u^{(n)} \right) \\
+ \alpha_2 u^{(n-1)} (u^{(n)} + u^{(n-1)} + u^{(n-2)}) - \alpha_3 u^{(n-1)} + \left( \frac{1}{2} \beta_3 \right) n \right\} \\
- \beta_2 \left[ u^{(n-1)} + (E - 1) ((n-1) u^{(n-1)}) \right] - \omega_3 (-1)^n = 0
\]

- Special case \( \beta_2 = 0 \) sums to fourth order \( dP_I \)
- Case \( \beta_2 \neq 0 \) sums to fourth order \( dP_{34} \) (generalized version)
- This fourth order \( dP_{34} \) related by new BT to complete known fourth order \( dP_{II} \)
Identifying discrete Painlevé hierarchies

We have extended our method of reduction of order to discrete equations; a reduction of order can also be effected by using properties of integrable lattices.

We consider systems of $N$ discrete equations in $p_n = (p_{1,n}, \ldots, p_{N,n})^T$ of the form

$$B[p_n]K[p_n] = 0$$

$B[p_n]$ is a Hamiltonian operator and we have the Miura map $p_n = M[u_n]$, $(u_n = (u_{1,n}, \ldots, u_{N,n})^T)$ for completely integrable lattice systems.

The differences with the continuous case:

(A) We allow the Hamiltonian operator of the modified system, $\tilde{B}[u_n]$, to depend on $u_n$. In particular, in the factorization

$$B[p_n] \bigg|_{p_n = M[u_n]} = M'[u_n] \tilde{B}[u_n] (M'[u_n])^\dagger,$$

we assume that

$$\tilde{B}[u_n] = C[u_n]DC[u_n],$$

and that it is through the constant-coefficient skew-adjoint matrix operator $D$ — which here plays the role played by $A \frac{d}{dx}$ in the differential case — that $\tilde{B}[u_n]$ depends on $E$. 

11
The summed modified system has then the form

\[ C[u_n] (M'[u_n])^† K[M[u_n]] + \omega(n) = 0. \]

(B) When we replace \( M[u_n] \) by \( p_n \), to form together with the Miura map the BT

\[
\begin{align*}
C[u_n] (M'[u_n])^† K[p_n] + \omega(n) &= 0 \\
p_n - M[u_n] &= 0
\end{align*}
\]

we assume that the product \( C[u_n] (M'[u_n])^† \) is such that the first part of the BT is a linear algebraic system for the modified variables \( u_n \) (perhaps after using the Miura map) having the unique solution

\[ u_n = G[p_n, \omega(n)] \]

We obtain a BT between the summed modified system and the summation of the original (unmodified) system

\[ p_n - M[G[p_n, \omega(n)]] = 0. \]

The reduction of order depends on \( D \).
A Toda example

We consider two-component systems in $p_n = (p_n, q_n)^T$ of the form

$$B[p_n]K[p_n] = 0$$

It arises as a reduction to discrete equations of a non-isospectral Toda lattice hierarchy.

$B[p_n]$ is the second of the three Hamiltonian operators of the Toda lattice hierarchy (Kupershmidt),

$$B[p_n] = \begin{pmatrix} E q_n - q_n E^{-1} & p_n (E - 1) q_n \\ q_n (1 - E^{-1}) p_n & q_n (E - E^{-1}) q_n \end{pmatrix} \quad E^k f_n = f_{n+k}$$

**Miura map**

$$p_n = M[u_n] = \begin{pmatrix} u_n + v_n \\ u_{n-1} v_n \end{pmatrix}, \quad u_n = (u_n, v_n)^T$$

**Factorization**

$$B[p_n] \bigg|_{p_n = M[u_n]} = M'[u_n] \tilde{B}[u_n] (M'[u_n])^\dagger$$

$M'[u_n]$ is the Fréchet derivative of the Miura map

$$M'[u_n] = \begin{pmatrix} 1 & 1 \\ v_n E^{-1} & u_{n-1} \end{pmatrix} \quad (M'[u_n])^\dagger = \begin{pmatrix} 1 & E v_n \\ 1 & u_{n-1} \end{pmatrix}$$

$\tilde{B}[u_n]$ Hamiltonian operator of the modified Toda lattice hierarchy
\[
\begin{pmatrix}
0 \\
v_n(1 - E^{-1})u_n \\
0
\end{pmatrix}
\]

that can be written in the form

\[
\begin{pmatrix}
0 \\
v_n(1 - E^{-1})u_n \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
v_n(1 - E^{-1})u_n \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
v_n(1 - E^{-1})u_n \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
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v_n(1 - E^{-1})u_n \\
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\end{pmatrix}
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\begin{pmatrix}
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v_n(1 - E^{-1})u_n \\
0
\end{pmatrix}
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\[
\begin{pmatrix}
0 \\
v_n(1 - E^{-1})u_n \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
v_n(1 - E^{-1})u_n \\
0
\end{pmatrix}
\]
It can be rewritten as

\[ u_n K_1[p_n] + q_{n+1} E K_2[p_n] + \omega_1 = 0 \]

\[ v_n K_1[p_n] + q_n K_2[p_n] + \omega_2 = 0 \]

\[ p_n - (u_n + v_n) = 0 \]

\[ q_n - u_{n-1} v_n = 0 \]

Linear algebraic system for the modified variables \( u_n = (u_n, v_n)^T \) with unique solution

\[ u_n = -\frac{q_{n+1} E K_2[p_n] + \omega_1}{K_1[p_n]}, \]

\[ v_n = -\frac{q_n K_2[p_n] + \omega_2}{K_1[p_n]} \]

Substituting \( u_n \) and \( v_n \) into the Miura map we obtain

- **Summed unmodified system**

\[ p_n K_1[p_n] + (E + 1) q_n K_2[p_n] + (\omega_1 + \omega_2) = 0 \]

\[ q_n \left( K_1[p_n] E^{-1} K_1[p_n] + p_n K_1[p_n] K_2[p_n] + q_{n+1} K_2[p_n] E K_2[p_n] \right) - \omega_1 \omega_2 = 0 \]

We have constructed a BT between the summed modified system and the summation of our original system.
Application: An asymmetric dP$_1$ hierarchy

We obtained the hierarchy of discrete equations

$$
\sum_{j=0}^{m} \gamma_{m-j-1} Q_n^j \left( \begin{array}{c} q_{n+1} - q_n \\ q_n(p_n - p_{n-1}) \end{array} \right) + \beta_{m-1} \left( \begin{array}{c} p_n \\ 2q_n \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
$$

\quad \text{(Gordo, Pickering and Zhu, 07)}

all $\gamma_k$ and $\beta_{m-1}$ are constant, $\beta_{m-1}$ being assumed nonzero, and where $Q_n$ is the recursion operator of the standard Toda lattice hierarchy

$$
Q_n = B[p_n] A^{-1}[p_n]
$$

with $A[p_n]$ given by

$$
A[p_n] = \left( \begin{array}{cc} 0 & (1 - E)q_n \\ q_n(E^{-1} - 1) & 0 \end{array} \right)
$$

The hierarchy can also be written in the form

$$
B[p_n] \left( \begin{array}{c} K_1[p_n] \\ K_2[p_n] \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
$$

$$
\left( \begin{array}{c} K_1[p_n] \\ K_2[p_n] \end{array} \right) = \sum_{j=0}^{m} \gamma_{m-j-1} L_{j+1,n} + \beta_{m-1} \left( \begin{array}{c} 0 \\ \frac{n-1}{q_n} \end{array} \right),
$$

Recursion relation

$$
B[p_n] L_{j,n} = A[p_n] L_{j+1,n} \quad j \geq 1
$$

$$
L_{1,n} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad L_{2,n} = \left( \begin{array}{c} -p_n \\ -1 \end{array} \right), \quad L_{3,n} = \left( \begin{array}{c} q_{n+1} + q_n + p_n^2 \\ p_n + p_{n-1} \end{array} \right)
$$
Each member of the hierarchy has been summed twice

Example: \( m = 1 \)

\[
\begin{pmatrix}
K_1[p_n] \\
K_2[p_n]
\end{pmatrix} = \gamma^{-1} \begin{pmatrix}
-p_n \\
-1
\end{pmatrix} + \gamma_0 \begin{pmatrix}
1 \\
0
\end{pmatrix} + \beta_0 \begin{pmatrix}
0 \\
\frac{1}{q_n}
\end{pmatrix}
\]

We obtain

\[
\gamma^{-1}(q_{n+1} + q_n + p_n^2) - \gamma_0 p_n - \beta_0(2n - 1) - a_1 - b_1 = 0
\]

\[
q_n(\gamma^{-1}p_n - \gamma_0)(\gamma^{-1}p_{n-1} - \gamma_0) - (\gamma^{-1}q_n - \beta_0(n - 1))^2 + (a_1 + b_1)(\gamma^{-1}q_n - \beta_0(n - 1)) - a_1 b_1 = 0
\]

Equivalent, under the BT

\[
p_n = u_n + v_n
\]
\[
q_n = u_{n-1}v_n
\]
\[
u_n = -\frac{\gamma^{-1}q_{n+1} - \beta_0 n - a_1}{\gamma^{-1}p_n - \gamma_0}
\]
\[
v_n = -\frac{\gamma^{-1}q_n - \beta_0(n - 1) - b_1}{\gamma^{-1}p_n - \gamma_0}
\]

to the system

\[
\gamma^{-1}u_n(u_n + v_{n+1} + v_n) - \gamma_0 u_n - \beta_0 n - a_1 = 0
\]
\[
\gamma^{-1}v_n(u_n + u_{n-1} + v_n) - \gamma_0 v_n - \beta_0(n - 1) - b_1 = 0
\]

the asymmetric \( dP_1 \) equation
A Volterra example

We consider one-component systems in $u_n$ of the form

$$ (\mathcal{B}[u_n] + c\mathcal{A}[u_n])K[u_n] = 0 $$

($u_n$ are the unmodified variables and $q_n$ the modified variables)

$\mathcal{B}[u_n]$ and $\mathcal{A}[u_n]$ are the Hamiltonian operators

$$ \mathcal{A}[u_n] = u_n(E^{-1} - E)u_n $$
$$ \mathcal{B}[u_n] = u_n(1 + E)(u_nE - E^{-1}u_n)(1 + E^{-1})u_n $$

$c \neq 0$ constant

It is the combination $\mathcal{B}[u_n] + c\mathcal{A}[u_n]$ which plays the role of $\mathcal{B}[p_n]$.

Under the **Miura map**

$$ u_n = M[q_n] = \frac{1}{4}c(1 - q_n)(1 + q_{n+1}), $$

we have the **factorization**

$$ (\mathcal{B}[u_n] + c\mathcal{A}[u_n])\Big|_{u_n = M[q_n]} = M'[q_n] \tilde{\mathcal{B}}[q_n] (M'[q_n])^\dagger $$

$$ M'[q_n] = \frac{1}{4}c((1 - q_n)E - (1 + q_{n+1})) $$
\( \tilde{\mathcal{B}}[q_n] \) is the Hamiltonian operator of the modified Volterra lattice hierarchy,

\[
\tilde{\mathcal{B}}[q_n] = -\frac{1}{4} c (q_n^2 - 1)(E - E^{-1})(q_n^2 - 1).
\]

The form of \( \mathcal{D} = -\frac{1}{4} c (E - E^{-1}) \) allows a double summation, with \( \omega(n) = -\gamma_1 - \gamma_2(-1)^n \), (\( \gamma_1 \) and \( \gamma_2 \) two arbitrary constants).

**THE RESULT**

- **Summed modified system**

\[
(q_n^2 - 1)\frac{1}{4} c (E^{-1}(1 - q_n) - (1 + q_{n+1})) K[M[q_n]] - \gamma_1 - \gamma_2(-1)^n = 0.
\]

- **The BT**

\[
(q_n - 1)u_{n-1}E^{-1}K[u_n] + (q_n + 1)u_nK[u_n] - \gamma_1 - \gamma_2(-1)^n = 0
\]

\[
u_n - \frac{1}{4} c (1 - q_n)(1 + q_{n+1}) = 0
\]

A linear algebraic equation for \( q_n \). Elimination of \( q_n \) gives

- **Summed unmodified system**

\[
u_n \left( u_n K[u_n] + u_{n+1}EK[u_n] \right) \left( u_n K[u_n] + u_{n-1}E^{-1}K[u_n] \right)
- \frac{1}{4} c \left( 2u_n K[u_n] - \gamma_1 - \gamma_2(-1)^n \right) \left( 2u_n K[u_n] + \gamma_1 - \gamma_2(-1)^n \right) = 0
\]
Application: A generalized $dP_{34}$ hierarchy

The hierarchy of integrable discrete equations

$$\sum_{j=0}^{m-1} \alpha_{m-j} R_n^j K_{1,n} + \beta_{m-1} R_n u_n + \beta_m u_n = 0,$$

$\beta_{m-1}$, $\beta_m$ and all $\alpha_k$ are constant

$$R_n = B[u_n] A^{-1}[u_n] = u_n (1 + E^{-1}) (u_n - u_{n+1} E^2) (E - 1)^{-1} u_n^{-1}$$

the recursion operator of the standard Volterra lattice hierarchy.

$$K_{1,n} = [u_n (u_{n-1} - u_{n+1})]$$

$$R_n u_n = u_n ((n-2) u_{n-1} - u_n - (n+1) u_{n+1}).$$

Setting $c = \tilde{\beta}_m = \beta_m / \beta_{m-1}$ we write the hierarchy as

$$\left( B[u_n] + \tilde{\beta}_m A[u_n] \right) K[u_n] = 0,$$

where

$$K[u_n] = \sum_{j=0}^{m-1} \alpha_{m-j} G_{j+1,n} + \beta_{m-1} \left[ \frac{1}{u_n} \left( \frac{1}{4} - \frac{n}{2} \right) \right]$$

and

$$G_{j+1,n} = \sum_{k=1}^{j+1} (-\tilde{\beta}_m)^{j+1-k} L_{k,n}.$$

$L_{j,n}$ satisfies the recurrence $B[u_n] L_{j,n} = A[u_n] L_{j+1,n}, j \geq 1$

$$L_{1,n} = -\frac{1}{2} \frac{1}{u_n}, \quad L_{2,n} = 1, \quad L_{3,n} = -(u_n + u_{n+1} + u_{n-1})$$
We set $\gamma_1 = a_m$, $\gamma_2 = b_m$, relabelled according to the position of the equation in the hierarchy.

Each member of the hierarchy has been summed twice

We obtain a BT between the summed modified hierarchy (a generalized dP$_{II}$ hierarchy which includes parity-dependent terms) and our original hierarchy summed twice (a generalized dP$_{34}$ hierarchy).

We have constructed (and identify) two discrete Painlevé hierarchies together with the BT relating them.

The simplest nontrivial example: $m = 2$

We start point

\[ K[u_n] = \alpha_1 G_{2,n} + \alpha_2 G_{1,n} + \beta_1 \left[ \frac{1}{u_n} \left( \frac{1}{4} - \frac{n}{2} \right) \right] \]

with

\[ G_{2,n} = L_{2,n} - \tilde{\beta}_2 L_{1,n} = 1 + \tilde{\beta}_2 \frac{1}{2u_n}, \quad G_{1,n} = L_{1,n} = -\frac{1}{2u_n} \]

\[ u_n K[u_n] = \alpha_1 u_n - \frac{1}{2} \beta_1 n + \gamma - \frac{1}{4} \beta_1, \]

and where

\[ \gamma = \frac{1}{2} \left( -\alpha_2 + \beta_1 + \alpha_1 \tilde{\beta}_2 \right). \]
The result for this example is the BT

\[
\begin{align*}
  u_n &= \frac{1}{4} \beta_2 (1 - q_n)(1 + q_{n+1}) \\
  q_n &= \frac{\alpha_1(u_{n-1} - u_n) + \frac{1}{2} \beta_1 + a_2 + b_2 (-1)^n}{\alpha_1(u_{n-1} + u_n) + 2\gamma - \beta_1 n}
\end{align*}
\]

between the \(dP_{II}\) equation

\[
\frac{1}{4} \alpha_1 \beta_2 (1 - q_n^2)(q_{n+1} + q_{n-1}) + 2\gamma q_n - \beta_1 n q_n - \frac{1}{2} \beta_1 - a_2 - b_2 (-1)^n = 0
\]

and the \(dP_{34}\) equation

\[
\begin{align*}
  u_n \left( \alpha_1(u_{n+1} + u_n) - \beta_1 (n + 1) + 2\gamma \right) \left( \alpha_1(u_n + u_{n-1}) - \beta_1 n + 2\gamma \right) \\
  -\frac{1}{4} \beta_2 \left[ (2\alpha_1 u_n - \beta_1 n + 2\gamma - \frac{1}{2} \beta_1 - b_2 (-1)^n)^2 - a_2^2 \right] &= 0
\end{align*}
\]
Conclusions

• A method of reduction of order for discrete systems having a certain structure. For both examples the reduction of order is by 2 and allows the identification of new Painlevé hierarchies.

• Characteristic features of integrable PDEs and lattices have implications for ODEs and discrete equations (e.g. Hamiltonian structures and BTs).

• What is important is the factorization of $B[u]$ and not the particular form of $L[u]$; non-integrable systems can be considered.

• Construct and identify new continuous and discrete Painlevé hierarchies.