Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients

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Stochastic heat equation

\[
\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sigma(X(t, x)) \dot{W}(x, t),
\]

\[
X(t, x) = \int p_t(x - y) X(0, y) dy + \int_0^t \int p_{t-s}(x - y) \sigma(X(s, y)) \dot{W}(dy, ds).
\]

where \( \dot{W} \) is the Gaussian noise

\[
E \left[ \dot{W}(x, t) \dot{W}(y, s) \right] = \delta(t - s) k(x - y).
\]

Main interest in the case of \( \dot{W} \) “white” noise in time and space in \( d = 1 \):

\[
k(z) = \delta(z).
\]
Uniqueness

We deal with the equation

\[ \frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sigma(X(t, x)) \dot{W}(x, t). \]

- **Pathwise uniqueness (PU):**
  \( X^1, X^2 \) — two solutions, \( X^1(0, \cdot) = X^2(0, \cdot) \)
  \implies X^1(t, \cdot) = X^2(t, \cdot), \forall t > 0.

- **Uniqueness in law (weak):**
  \( X^1, X^2 \) — two solutions (even on different spaces),
  \( X^1(0, \cdot) = X^2(0, \cdot) \implies \{X^1(t, \cdot)\}_{t \geq 0} \overset{\text{law}}{=} \{X^2(t, \cdot)\}_{t \geq 0}. \)
Uniqueness

\[
\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sigma(X(t, x)) \dot{W}(x, t).
\]

If \( \dot{W} \) is a space-time white noise, then function-valued solution exists if \( d = 1 \).

Uniqueness?
\( \sigma \) — Lipschitz \( \implies \) PU follows easily.
\( \sigma \) - non-Lipschitz ?
Super-Brownian motion

Branching Brownian motions in $\mathbb{R}^d$.

$X^n$:

$\sim n$ particles in $\mathbb{R}^d$ at time 0.

$\frac{1}{n}, \frac{2}{n}, \ldots$ — times of death or split,

$p_0 = p_2 = \frac{1}{2}$ — probabilities of death or split.

Critical branching: mean number of offspring $= 1$.

New particles move as independent Brownian motions.

\[
X^n_t(A) = \frac{\# \text{ particles in } A \text{ at time } t}{n}, \quad A \subset \mathbb{R}^d.
\]

$X^n \Rightarrow X$,

$X$ is a super-Brownian motion — measure-valued process.
Properties of SBM

- Singular measure if $d > 1$. 

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Properties of SBM

- Singular measure if $d > 1$.

- Existence of density only in $d = 1$:
  
  \[ X_t(dx) = X_t(x)dx \]
Properties of SBM

- Singular measure if \( d > 1 \).

- Existence of density only in \( d = 1 \):
  \[ X_t(dx) = X_t(x)dx \]

- \( d = 1 \). \( X_t(x) \) is jointly continuous in \((t, x)\). N. Konno, T. Shiga(88); M. Reimers (89):
  \[ \frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sqrt{X} \dot{W}. \]
  \( \dot{W} \) — Gaussian space-time white noise.
Uniqueness for SBM

\[
\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sqrt{X} \dot{W}.
\]

**Weak uniqueness** (in law) holds. Follows by duality method.

Notation: \( \psi(f) = \int_{\mathbb{R}^d} \psi(x)f(x) \, dx \).

\[
E \left[ e^{-X_t(\phi)} \right] = e^{-X_0(V_t)}, \quad \forall t \geq 0, \phi \geq 0.
\]

where

\[
\begin{align*}
\frac{\partial V}{\partial t} &= \frac{1}{2} \Delta V - \frac{1}{2} V^2, \\
V_0 &= \phi.
\end{align*}
\]

**Pathwise uniqueness (PU)?**
\( \sqrt{X} \) — non-Lipschitz.
Is there a chance to get **PU**?
Pathwise uniqueness for SDEs

\[ dX_t = \sigma(X_t) dB_t \]

\( B_t \) is a one-dimensional Brownian motion.

**Theorem 1 (Yamada, Watanabe (71))**

*If \( \sigma \) is Hölder continuous with exponent 1/2, then PU holds.*

**Remark 2**

*There are counter examples for \( \sigma \) which is Hölder continuous with exponent less than 1/2.*
Proof of Theorem 1

Define (in a special way) function \( \phi_n \in C_c^\infty(R) \) s.t.
\[
\phi_n(x) \to |x|, \quad \phi_n'' \to \delta_0, \quad \text{as } n \to \infty.
\]
Define \( \tilde{X} = X^1 - X^2 \).
Then \( \tilde{X}_0 = 0 \) and
\[
d\tilde{X}_t = (\sigma(X^1_t) - \sigma(X^1_t))dB_t.
\]

Ito’s formula:
\[
\phi_n(\tilde{X}_t) = \int_0^t \phi_n'(\tilde{X}_s)(\sigma(X^1_s) - \sigma(X^2_s)) dB_s
\]
\[
+ \frac{1}{2} \int_0^t \phi_n''(\tilde{X}_s)(\sigma(X^1_s) - \sigma(X^2_s))^2 ds
\]

By the choice of \( \phi_n \) and Hölder assumptions on \( \sigma \) one can show
\[
E\left[\phi_n(\tilde{X}_t)\right] \leq cE\left[\int_0^t \phi_n''(\tilde{X}_s)|\tilde{X}_s| ds\right]
\]
\[
\to 0, \quad \text{as } n \to \infty.
\]
**SPDE driven by colored noise**

**Non-uniqueness**

Back to SPDEs

- **SPDE for super-Brownian motion density in** $d = 1$
  
  $\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + \sqrt{X(t, x)} \dot{W}(x, t)$.

  $\dot{W}$ — space-time white noise.

  Numerous attempts to prove **PU** failed.

  **PU** question is still open.

- **General stochastic heat equation**

  Let $\sigma(x)$ be Hölder continuous with exponent $\gamma$.

  Our main interest: conditions on $\gamma$ such that **PU** holds for

  $\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W}$,

  where $\dot{W}$ is space-time white noise.
Theorem 3 (Perkins, M., 09)

Let \( \sigma(x) \) be Hölder continuous with exponent \( \gamma \).
For any \( \gamma > 3/4 \), PU holds for

\[
\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W},
\]

where \( \dot{W} \) is space-time white noise.
We start with the equations that are close to the above. Take less singular (spatially) noise. Consider the problem of $PU$ for

$$\frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W},$$

where the noise $\dot{W}$ is “white” in time and “colored” in space:

$$E[\dot{W}(x, t) \dot{W}(y, s)] = \delta(t - s)k(x - y).$$

**Assumptions**

*(H$(\alpha)$)* $k(z) \leq |z|^{-\alpha}$, $0 \leq \alpha < d$.

*(H$(\gamma)$)* $\sigma(x)$ is Hölder cont. with exponent $\gamma$.

**Existence of function-valued solution:**

$0 \leq \alpha < 2 \wedge d$, Peszat-Zabczyk(00), Dalang(99) (for Lipschitz case. Similar for non-Lipschitz).
Uniqueness for SPDE driven by colored noise

Theorem 4 (Sturm, Perkins, M., 05)

$PU$ holds if

$$\alpha < 2\gamma - 1.$$ 

Remark 5

For $d = 1$, $\alpha = 1$ (white noise case) we have

$$\gamma > 1 \ldots$$
Proof of Theorem 4

\[ \frac{\partial X}{\partial t} = \frac{1}{2} \Delta X + \sigma(X) \dot{W}, \]

\( X^1, X^2 \) — two solutions, \( \tilde{X} = X^1 - X^2. \)

\[ \frac{\partial \tilde{X}_t(x)}{\partial t} = \frac{1}{2} \Delta \tilde{X}_t(x) + (\sigma(X^1_t(x)) - \sigma(X^2_t(x))) \dot{W}(t, x). \]

Choose the functions \( \phi_n, f^n: \)

\[ \phi_n(x) \to |x|, \quad \phi''_n \to \delta_0, \quad \text{as} \quad n \to \infty. \]

\[ f^n_x \to \delta_x, \quad \text{as} \quad n \to \infty. \]

Apply Ito,

\[ E \left[ \phi_n(\tilde{X}_t(f^n_x)) \right] = \ldots \]

\[ \downarrow \quad \downarrow \quad \text{(wish)} \]

\[ E \left[ |\tilde{X}_t(x)| \right] = 0. \]
One needs

\[ l^n(t, x) = E \left[ \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \phi''_n(|\tilde{X}_s(f^n_x)|) |\tilde{X}_s(z)|^\gamma |\tilde{X}_s(y)|^\gamma \times f^n_x(z)f^n_x(y)k(z - y)dz dy ds \right] \rightarrow 0. \]
One needs

\[ I^n(t, x) = E \left[ \frac{1}{2} \int_0^t \int_{R^{2d}} \phi''(|\tilde{X}_s(f^n_x)|) |\tilde{X}_s(z)|^\gamma |\tilde{X}_s(y)|^\gamma \times f^n_x(z)f^n_x(y)k(z - y)dz dy ds \right] \rightarrow 0. \]

Crucial: Hölder exponent of \( \tilde{X}_s(x) \) in \( x \).
One needs

\[ I^n(t, x) = E \left[ \frac{1}{2} \int_0^t \int_{R^{2d}} \phi''(|\tilde{X}_s(f^n_x)|)|\tilde{X}_s(z)|^{\gamma} |\tilde{X}_s(y)|^{\gamma} \right. \\
\left. \times f^n_x(z)f^n_x(y) k(z - y) dz dy ds \right] \to 0. \]

Crucial: Hölder exponent of \( \tilde{X}_s(x) \) in \( x \).
Suppose \( \tilde{X} \) is \( \xi \)-Hölder continuous. Then we can show:

\[ I^n \to 0, \]

if

\[ \frac{\alpha}{\xi} < (2\gamma - 1), \]

or

\[ \gamma > \frac{1}{2} + \frac{\alpha}{2\xi}, \]
We got condition for PU:

\[ \alpha < \xi (2\gamma - 1). \]

**Proposition 6 (Sanz-Solé, Sarrà)**

*For any \( \xi < 1 - \frac{\alpha}{2} \), \( \tilde{X}_s(\cdot) \) is Hölder continuous with exponent \( \xi \).*

By Theorem of Sanz-Solé, Sarrà we get

\[ \alpha < \frac{2\gamma - 1}{\gamma + 1/2}. \]

**Bad:** \( \gamma \nearrow 1 \implies \alpha < 2/3. \)
Proposition 7 (Sturm, Perkins, M.)

At the points $x$ where $\tilde{X}_s(x) = 0$, $\tilde{X}_s(\cdot)$ is $\xi$-Hölder continuous

$$\forall \xi < \frac{1 - \alpha/2}{1 - \gamma} \wedge 1.$$  

Note that for $\alpha < 2\gamma - 1$,

$$\frac{1 - \alpha/2}{1 - \gamma} \geq 1.$$  

Hence we have

Corollary 8 (Sturm, Perkins, M.)

Let $\alpha < 2\gamma - 1$. At the points $x$ where $\tilde{X}_s(x) = 0$, $\tilde{X}_s(\cdot)$ is $\xi$-Hölder continuous

$$\forall \xi < 1.$$  

Remark Mueller-Tribe have the result similar to Proposition 7.
By condition on PU ($\alpha < \xi(2\gamma - 1)$) we get

$$\alpha < 2\gamma - 1$$

and this finishes the proof of Theorem 4.
Optimality of the bound

\[ \alpha < 2\gamma - 1 \ ? \]

Note: \( d = 1, \alpha = 1 \) (white noise case) gives

\[ \gamma > 1 \ldots \]
Theorem 3

If one replaces the condition

$$\forall \xi < \frac{1 - \alpha/2}{1 - \gamma} \wedge 1$$

by

$$\forall \xi < \frac{1 - \alpha/2}{1 - \gamma}.$$  

(allowing $\xi$ to be $> 1$)

then in $d = 1$ one gets the following condition on $\mathbf{PU}$:

$$\alpha < 2(2\gamma - 1) \quad \text{or} \quad \gamma > \frac{\alpha}{4} + \frac{1}{2},$$

That is for $\alpha = 1$ (white noise case in $d = 1$) we get

$$\gamma > \frac{3}{4}$$

— Theorem 3!!
Non-uniqueness

▶ Is 3/4 sharp?
Counter example: for $\gamma < 3/4$ try to construct non-trivial solution to
\[
\begin{cases}
\frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) + |X(t, x)|^{\gamma} \dot{W}(x, t), \\
X(0, \cdot) = 0.
\end{cases}
\]
Non-uniqueness

- **Is 3/4 sharp?**
  Counter example: for \( \gamma < 3/4 \) try to construct non-trivial solution to
  \[
  \begin{align*}
  \frac{\partial}{\partial t} X(t, x) &= \frac{1}{2} \Delta X(t, x) + |X(t, x)|^{\gamma} \dot{W}(x, t), \\
  X(0, \cdot) &= 0.
  \end{align*}
  \]

- **Hard killing model** \((X^1, X^2)\):
  \[
  \begin{align*}
  \frac{\partial}{\partial t} X^i(t, x) &= \frac{1}{2} \Delta X^i(t, x) + X^i(t, x)^{\gamma} \dot{W}^i(x, t) \\
  &\quad - \frac{\partial A}{\partial t} + \frac{\partial I^i}{\partial t}, \quad i = 1, 2, \\
  X^i &\geq 0, \quad X^1 X^2 = 0.
  \end{align*}
  \]

  \(I^i(dt, dx)\) is immigration of mass,
  \(A\) is the killing term,
  \(W^1, W^2\) are independent.
  If \(I^1 = I^2\), then \(X \equiv X^1 - X^2\) solves (1).
Aim: construct non-trivial \((X^1, X^2)\).

Approximation: Let \(\eta^{1, \varepsilon}, \eta^{2, \varepsilon}\) be independent Poisson random measures on \(R_+ \times [-1, 1]\) with intensity \(\varepsilon^{-1} dt \, dx\).

\[
I^{i, \varepsilon}(t, A) \equiv \varepsilon \eta^{i, \varepsilon}([0, t] \times A), \ i = 1, 2.
\]

\((X^{1, \varepsilon}, X^{2, \varepsilon})\) is the corresponding hard killing process.

Clearly as \(\varepsilon \downarrow 0\),

\[
I^{i, \varepsilon} \Rightarrow dt \, dx 1(x \in [-1, 1]), \ i = 1, 2.
\]

Hence

\[
X^{\varepsilon} = X^{1, \varepsilon} - X^{2, \varepsilon}
\]

\[
\Rightarrow X, \ \text{as} \ \varepsilon \downarrow 0.
\]

where \(X\) solves (1).

We would like to show that \(X\) is non-trivial for

\[
\gamma < \frac{3}{4}.
\]
Representation

\[
X_t^{i,\varepsilon} = \sum_{k: t_k \leq t} X_t^{i,\varepsilon,k}
\]

where \(X_t^{i,\varepsilon,k}\) is the "cluster" starting at the atom \(\varepsilon \delta_{x_k,t_k}\) of the immigration measure \(I_t^{i,\varepsilon}\).

Consider \(X^{1,\varepsilon}\). In the absence of the killing \((A = 0)\),

\[
\inf_{\varepsilon} P(\exists \text{ a cluster starting at some } t_k \leq 1/2 \text{ and surviving until } t = 1) > 0.
\]

Let \(Y^{1,\varepsilon}\) be one of such clusters. Shift time and space so that it starts at \(\delta_{0,0}\).
It could be checked that for small $t$ the total mass of the surviving cluster evolves as

$$\left\langle Y_{1,\varepsilon}^1, 1 \right\rangle \sim t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}} \gg t^{3/2}, \text{ for } \gamma < 3/4.$$ 

Most of the mass of $\{Y_{s,\varepsilon}^1, s \leq t\}$ is inside the parabola

$$B_t = \{(s, x) : |x| \leq \sqrt{s}, s \leq t\}$$

How $Y_{1,\varepsilon}^1$ could be killed by $X_{2,\varepsilon}^2$ before (small) time $t$ with probability 1?

- By clusters of $X_{2,\varepsilon}^2$ born before time 0 at $[-t, 0]$ for $t$ small.
- By clusters of $X_{2,\varepsilon}^2$ born after time 0 inside $B_t$. 

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Killing by clusters of $X^{2,\varepsilon}$ born at $[-t, 0]$ for $t$ small.

For $t$ small, to touch $(0, 0)$ any cluster of $X^{2,\varepsilon}$ born at $[-t, 0]$ should be born inside

$$\tilde{B}_t = \{(s, x) : |x| \leq \sqrt{|s|}, s \geq -t\}$$

One can show that by "branching processes" argument that to survive $t$ units of time the immigration of mass to $X^{2,\varepsilon}$ inside $\tilde{B}_t$ should be at least of order $t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}}$. However the immigration is just of order

$$t^{3/2} \ll t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}}, \gamma < 3/4,$$

and hence the mass of $X^{2,\varepsilon}$ that was born inside $\tilde{B}_t$ dies out by time 0.
Killing by clusters of $X^{2,\varepsilon}$ born after time 0 inside $B_t$.

The immigration of mass in $X^{2,\varepsilon}$ inside $B_t$ is of order $t^{3/2}$. The mass of $Y^{1,\varepsilon}$ at time $t$ is of order

$$t^{\frac{1}{4(1-\gamma)} + \frac{1}{2}} \gg t^{3/2}$$

and hence $Y^{1,\varepsilon}$ "wins" the competition.