SIMULATION-BASED COMPUTATION OF THE WORKLOAD CORRELATION FUNCTION IN A LÉVY-DRIVEN QUEUE

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Joint work with Peter Glynn (Stanford)
THE BOTTOMLINE
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There are things that are more urgent than math!
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SOME TRUTHS

It is true that Brazil won the World Cup five times...
SOME TRUTHS

It is true that Brazil won the World Cup five times...

... but Holland should have won it *at least* five times!!!
WHAT WENT WRONG, PART I: 1974
WHAT WENT WRONG, PART I: 1974
WHAT WENT WRONG, PART II: 1978
WHAT WENT WRONG, PART II: 1978
WHAT WENT WRONG, PART III: 1990
WHAT WENT WRONG, PART III: 1990
WHAT WENT WRONG, PART IV: 1994
WHAT WENT WRONG, PART IV: 1994
WHAT WENT WRONG, PART V: 1998
WHAT WENT WRONG, PART V: 1998
WHAT WENT WRONG, PART VI: 2002
WHAT WENT WRONG, PART VI: 2002
WHAT WENT WRONG, PART VI: 2002
WHAT WENT WRONG, PART VII: 2006
WHAT WENT WRONG, PART VII: 2006
MORE EMBARRASING NEWS
MORE EMBARRASING NEWS

This is only my second RESIM workshop...
AN ADVERTISEMENT

Book by Gerardo Rubino and Bruno Tuffin.
THIS TALK

So far: estimation of rare-event probabilities...
THIS TALK

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Now: estimation of moments that vanish in some parameter —
leading example: correlations that vanish when the time lag grows...
So far: estimation of rare-event probabilities...

Now: estimation of moments that vanish in some parameter —
leading example: correlations that vanish when the time lag grows...

We do so in the context of a Lévy driven queue.
LÉVY-DRIVEN QUEUE?

‘Natural notion’ in discrete time.

Let $Q_0 = x$. Then

$$Q_{n+1} = \max\{Q_n + Y_n, 0\}.$$
LÉVY-DRIVEN QUEUE?

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Let $Q_0 = x$. Then

$$Q_{n+1} = \max\{Q_n + Y_n, 0\}.$$ 

Iterate: $Q_{n+1} = \max\{Q_{n-1} + Y_{n-1} + Y_n, Y_n, 0\}$. With $X_n := \sum_{i=0}^{n} Y_i$, this leads to

$$Q_n = X_n + \max\left\{ x, \max_{0 \leq i \leq n} -X_i \right\}.$$
LÉVY-DRIVEN QUEUE?

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$$Q_n = X_n + \max \left\{ x, \max_{0 \leq i \leq n} -X_i \right\}.$$  

Then take the continuous-time counterpart:

$$Q_t = X_t + \max\{x, L_t\}, \ t \geq 0,$$

with

$$L_t := \sup_{0 \leq u \leq t} -X_u = -\inf_{0 \leq u \leq t} X_u.$$
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(also called ‘reflected Brownian motion’).
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(also called ‘reflected Brownian motion’).

Queue with stationary increments: \( X_t - X_{t-s} \) has, for a given \( s \), the same distribution, irrespective of \( t \).

Steady-state distribution \( Q \equiv \lim_{t \to \infty} Q_t \) exists if \( \mathbb{E}X_1 < 0 \).
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Steady-state distribution $Q \equiv \lim_{t \to \infty} Q_t$ exists if $\mathbb{E}X_1 < 0$.

$(Q_t)_t$ often referred to as *workload process*. 
DEPENDENCE STRUCTURE

What happens if we feed the queue by a highly correlated input process $(X_t)_t$?
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Is this high correlation inherited by the workload process?

**Key question**: suppose input process is long-range dependent — what can be said about

\[
r(t) := \text{Cov} (Q_0, Q_t)\
\]

*Is the workload process long-range dependent?*
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*Is the workload process long-range dependent?*

**Conjecture** (M., Norros, & Glynn, *Ann. Appl. Prob. ’09*): let \((X_t)_t\) be fractional Brownian motion. Then,

\[ r(t) \sim t^{2H-2}; \]

that is, workload process is long-range dependent for \(H \in (\frac{1}{2}, 1)\).
DEPENDENCE STRUCTURE

Was not proven yet.

Main difficulty: not straightforward to write $\text{Cov}(Q_0, Q_t)$ as large-deviation probability.

Therefore: other techniques are needed.
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Therefore: other techniques are needed.

Classical importance sampling techniques do not work!

How to efficiently simulate small correlations?
THIS TALK

Same question, but now the queue is driven by a Lévy process (stationary and independent increments).
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Now input process is (evidently) not long-range dependent...
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Now input process is (evidently) *not* long-range dependent...

... but what can be said about workload process?
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Now input process is (evidently) *not* long-range dependent...

... but what can be said about workload process?

Focus on

★ Structural results;
★ Efficient simulation techniques.
SPECTRALLY ONE-SIDED LÉVY PROCESSES

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$(X_t)_t$: Lévy process without one-sided jumps; drift $\mathbb{E} X_1 < 0$.

Key object in applied probability.
SPECTRALLY ONE-SIDED LÉVY PROCESSES

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$(X_t)_t$: Lévy process without one-sided jumps; drift $\mathbb{E}X_1 < 0$.

Key object in applied probability.

Spectrally one-sided: jumps are either only positive (spectrally positive)

or only negative (spectrally negative).
SPECTRALLY ONE-SIDED LÉVY PROCESSES: EXAMPLES

Important examples of spectrally one-sided Lévy processes are:

- *Brownian motion with drift* (being actually both spectrally positive and negative).

- *Compound Poisson with drift*. Non-negative jobs arrive according to a Poisson process of rate \( \lambda \); the jobs \( B_1, B_2, \ldots \) are i.i.d. samples from a distribution with Laplace transform \( b(\alpha) := \mathbb{E}e^{-\alpha B} \); the storage system is continuously depleted at a rate 1.

  If drift would be positive, and jobs would be i.i.d. samples from a non-positive distribution (i.e., the jumps are downward), the process is spectrally negative.
INDUCED QUEUEING PROCESS

$(Q_t)_t$ denotes the *reflection* of spectrally one-sided Lévy process $(X_t)_t$ at $0$:

$$Q_t := X_t + \max\{Q_0, -M_t\}, \quad t \geq 0,$$

where $M = (M_t)_{t \geq 0}$ is the decreasing process defined by $M_t := \inf_{0 \leq u \leq t} X_u$.

We assume that the queue is already in stationarity at time $t = 0$. 
SPECTRALLY ONE-SIDED

Spectrally-positive case:

Laplace exponent: \( \varphi \). Thus \( \varphi(\alpha) := \log \mathbb{E}e^{-\alpha X_1} \).
Increasing and convex on \([0, \infty)\), with slope \(-\mathbb{E}X_1\) in the origin.
Inverse is \( \psi \).

Assume that \( X_t \) is not a subordinator, i.e., a monotone process.
SPECTRALLY ONE-SIDED

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Laplace exponent: \( \varphi \). Thus \( \varphi(\alpha) := \log \mathbb{E} e^{-\alpha X_1} \).
Increasing and convex on \([0, \infty)\), with slope \(-\mathbb{E} X_1\) in the origin.
Inverse is \( \psi \).

Stationary behavior is well understood;

the steady-state distribution of \( Q := \lim_{t \to \infty} Q_t \) is given through

\[
\kappa(s) := \mathbb{E} e^{-sQ} = s \frac{\varphi'(0)}{\varphi(s)}, \quad s \geq 0.
\]

Generalized Pollaczek-Khinchine formula.
SPECTRALLY ONE-SIDED

Spectrally-positive case, ctd.:

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*Generalized Pollaczek-Khinchine formula.*

Mean and variance:

$$\mu := \mathbb{E}Q = \frac{\varphi''(0)}{2\varphi'(0)}; \quad \nu := \text{Var}Q = \left(\frac{\varphi''(0)}{2\varphi'(0)}\right)^2 - \frac{\varphi'''(0)}{3\varphi'(0)}.$$  

We assume that $\nu < \infty$, i.e., $\varphi'''(0) < \infty$.

Brownian motion: $\varphi(\alpha) = -\alpha\mu + \frac{1}{2}\alpha^2\sigma^2$;

Compound Poisson: $\varphi(\alpha) = \alpha - \lambda + \lambda b(\alpha)$.  

SPECTRALLY ONE-SIDED

Spectrally-negative case:

Define $\Phi(\beta) := \log \mathbb{E} e^{\beta X_1}$, for $\beta \geq 0$.

Again rule out that $X_t$ is a subordinator (and recalling that $\Phi'(0) = \mathbb{E} X_1 < 0$).

Note: $\Phi(\beta)$ is no bijection on $[0, \infty)$. Therefore define the right inverse through

$$\Psi(q) := \sup\{\beta \geq 0 : \Phi(\beta) = q\}.$$

Realize that $\beta_0 := \Psi(0) > 0$. 
SPECTRALLY ONE-SIDED

Spectrally-negative case, ctd.:

$e^{\beta_0 X_t}$ is martingale, with $\beta_0 := \Psi(0) > 0$.

‘Optional sampling’ thus gives, for any positive $x$, $\mathbb{P}(\exists t \geq 0 : X_t > x)e^{\beta_0 x} = 1$. 
SPECTRALLY ONE-SIDED

Spectrally-negative case, ctd.:

\( e^{\beta_0 X_t} \) is martingale, with \( \beta_0 := \Psi(0) > 0 \).

‘Optional sampling’ thus gives, for any positive \( x \), \( \mathbb{P}(\exists t \geq 0 : X_t > x) e^{\beta_0 x} = 1 \).

Now \( Q \) is distributed as the supremum over \( t \geq 0 \) of \( X_t \) (‘Reich’s identity’).

Hence: \( Q \) is exponentially distributed with mean \( 1/\beta_0 \).

It follows that \( v = 1/\beta_0^2 \).
CORRELATION FUNCTION

This rest of this talk focuses on

★ structural properties of the workload covariance function;
★ efficient computation of the workload covariance function, through simulation.
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⋆ structural properties of the workload covariance function;
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Recall: correlation function is

\[ r(t) := \frac{\Cov(Q_0, Q_t)}{\sqrt{\text{Var}Q_0 \text{Var}Q_t}} = \frac{\mathbb{E}(Q_0Q_t) - (\mathbb{E}Q_0)^2}{\text{Var}Q_0}. \]

Its Laplace transform is

\[ \rho(\vartheta) = \int_0^\infty e^{-\vartheta t} r(t) \, dt. \]
**CORRELATION FUNCTION**

**Theorem:** In the spectrally-positive case, the Laplace transform $\rho(\vartheta)$ of $r(t)$ is given by

$$
\rho(\vartheta) = \frac{1}{\vartheta} - \frac{\varphi''(0)}{2v\vartheta^2} + \frac{\varphi'(0)}{v\vartheta^2} \left( \frac{1}{\vartheta \psi'(\vartheta)} - \frac{1}{\psi(\vartheta)} \right).
$$

CORRELATION FUNCTION, ctd.

- For $T \sim \text{Exp}(\vartheta)$ independent of $X_t$ we have (Kella, Boxma, and M., J. Appl. Prob., 2006)

$$\mathbb{E}(e^{-sQ_T} | Q_0 = q) = \frac{\vartheta}{\vartheta - \varphi(s)} \left( e^{-sq} - s \frac{e^{-\psi(\vartheta)q}}{\psi(\vartheta)} \right). \quad (1)$$

- Differentiate (1) w.r.t. $s$ and let $s \downarrow 0$ yields

$$\int_0^\infty \vartheta e^{-\vartheta t} \mathbb{E}(Q_t | Q_0 = q) dt = -\frac{\varphi'(0)}{\vartheta} + q + \frac{e^{-\psi(\vartheta)q}}{\psi(\vartheta)}.$$

- The result follows from

$$\int_0^\infty e^{-\vartheta t} \mathbb{E}(Q_0Q_t) dt = \int_0^\infty \frac{q}{\vartheta} \int_0^\infty \vartheta e^{-\vartheta t} \mathbb{E}(Q_t | Q_0 = q) dt \, d\mathbb{P}(Q_0 \leq q).$$
Example: Brownian motion

\[ \rho(\vartheta) = \frac{1}{\vartheta} - \frac{2}{\vartheta^2} + \frac{2}{\vartheta^3} \left( \sqrt{1 + 2\vartheta} - 1 \right) \]

which we can explicitly invert to obtain

\[ r(t) = 2(1 - 2t - t^2) \left( 1 - \Phi_N(\sqrt{t}) \right) + 2\sqrt{t}(1 + t)\phi_N(\sqrt{t}), \]

where \( \Phi_N(\cdot) \) (resp. \( \phi_N(\cdot) \)) is the standard Normal distribution (resp. density).
CORRELATION FUNCTION, ctd.

Es-Saghouani and M.: structural properties for spectrally-positive case, such as $r(t)$ is positive, decreasing, convex, complementing M/G/1 results by Ott.

Relying on machinery of completely monotone functions (Bernstein, 1929). (We’ll demonstrate this concept for the spectrally-negative case.)

In addition: asymptotics of $r(t)$ for $t$ large.
CORRELATION FUNCTION

But how about spectrally-negative case?
CORRELATION FUNCTION

But how about spectrally-negative case?

$q$-scale functions: $W^{(q)}(x)$ is a strictly increasing and continuous function whose Laplace transform satisfies

$$
\int_{0}^{\infty} e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\Phi(\beta) - q}, \quad \beta > \Psi(q),
$$

and in addition

$$
Z^{(q)}(x) := 1 + q + \int_{0}^{x} W^{(q)}(y) dy.
$$
CORRELATION FUNCTION: TRANSFORM

Pistorius, *J. Th. Prob.*, 2004: transform (with respect to $t$) of the density of $Q_t$, given that $Q_0 = x$:

$$\int_0^\infty e^{-qt} \mathbb{P}_x(Q_t = y) dt = e^{-\Psi(q)y} \frac{\Psi(q)}{q} Z^{(q)}(x) - W^{(q)}(x - y).$$
Pistorius, J. Th. Prob., 2004: transform (with respect to $t$) of the density of $Q_t$, given that $Q_0 = x$:
\[
\int_0^\infty e^{-q^t} \mathbb{P}_x(Q_t = y) dt = e^{-\Psi(q)y} \frac{\Psi(q)}{q} Z^{(q)}(x) - W^{(q)}(x - y).
\]

Hence, with $T$ an exponential random variable with mean $q^{-1}$,
\[
\int_0^\infty e^{-\beta x} \mathbb{E}_x e^{-\alpha Q_T} dx = I_1 - I_2;
\]

here
\[
I_1 := \int_0^\infty \int_0^\infty q e^{-\beta x} e^{-\alpha y} e^{-\Psi(q)y} \frac{\Psi(q)}{q} Z^{(q)}(x) dx dy = \ldots = \frac{\Psi(q)}{\Psi(q) + \alpha \beta} \left( 1 + \frac{q}{\Phi(\beta) - q} \right),
\]
\[
I_2 := \int_0^\infty \int_0^\infty q e^{-\beta x} e^{-\alpha y} W^{(q)}(x) dx dy = \ldots = \frac{q}{\alpha + \beta \Phi(\beta) - q}.
\]
So we have an expression for
\[ \int_{0}^{\infty} \beta e^{-\beta x} E_x e^{-\alpha Q_T} \, dx. \]
CORRELATION FUNCTION: TRANSFORM

So we have an expression for
\[
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\]

Sanity checks:

- Plugging in \( \alpha = 0 \) yields 1.
- Plugging in \( \beta = \beta_0 \) yields the steady-state transform \( \beta_0/(\beta_0 + \alpha) \):
  
  when starting in the queue’s equilibrium distribution at time 0, the workload is still in stationarity after an exponentially distributed time (irrespectively of \( q \)).
CORRELATION FUNCTION: TRANSFORM

With $T$ having an exponential distribution with mean $q^{-1}$,

$$\int_0^\infty q e^{-qt} \mathbb{E}(Q_0 Q_t) dt = \int_0^\infty \beta_0 x e^{-\beta_0 x} \mathbb{E}_x Q_T dx = \lim_{\alpha \downarrow 0} \int_0^\infty e^{-\beta x} \mathbb{E}_x e^{-\alpha Q_T} dx |_{\beta=\beta_0}.$$
CORRELATION FUNCTION: TRANSFORM

With $T$ having an exponential distribution with mean $q^{-1}$,

$$\int_0^\infty q e^{-qt} \mathbb{E}(Q_0Q_t) dt = \int_0^\infty \beta_0 x e^{-\beta_0 x} \mathbb{E}_x Q_T dx = \lim_{\alpha \downarrow 0} \frac{d}{d\alpha} \left[ \beta \cdot \frac{d}{d\beta} \int_0^\infty e^{-\beta x} \mathbb{E}_x e^{-\alpha Q_T} dx \bigg|_{\beta=\beta_0} \right].$$

We eventually find, after considerable calculus, the following result.

**Theorem:** In the spectrally-negative case, the Laplace transform $\rho(\psi)$ of $r(t)$ is given by

$$\rho(q) := \int_0^\infty r(t) e^{-qt} dt = \frac{1}{q} + \frac{\beta_0^2}{q^2} \Phi'(\beta_0) \left( \frac{1}{\Psi(q)} - \frac{1}{\beta_0} \right).$$
CORRELATION FUNCTION: TRANSFORM

**Corollary:** For the spectrally-negative case,

\[ \rho(0) := \int_0^\infty r(t)dt = \frac{1}{\beta_0 \Phi'((\beta_0)} + \frac{\Phi''((\beta_0)}{2(\Phi(\beta_0))^{3}} < \infty. \]

This is not true in the spectrally-positive case.
CORRELATION FUNCTION: STRUCTURAL PROPERTIES

**Theorem:** $r(\cdot)$ is positive, decreasing, and convex.

*Proof:* Mimic the proof in Es-Saghouani and M. for the spectrally-positive case. $C$: class of completely monotone functions.
Integration by parts:

\[ \rho^{(1)}(q) := \int_0^\infty r'(t) e^{-qt} dt = \frac{\beta_0^2}{q} \Phi'(\beta_0) \left( \frac{1}{\Psi(q)} - \frac{1}{\beta_0} \right); \]

\[ \rho^{(2)}(q) := \int_0^\infty r''(t) e^{-qt} dt = -r'(0) + \beta_0^2 \Phi'(\beta_0) \left( \frac{1}{\Psi(q)} - \frac{1}{\beta_0} \right). \]
CORRELATION FUNCTION: STRUCTURAL PROPERTIES

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Later we show that \( \Psi(0)/\Psi(q) \in \mathcal{C} \).

Conclude: \( \rho^{(2)}(q) \) is in \( \mathcal{C} \), and hence \( r''(\cdot) \) is positive, i.e., \( r(\cdot) \) is convex.
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Known: \( f(q) \in \mathcal{C} \) implies that, with \( g(q) := (f(0) - f(q))/q \), also \( g(q) \in \mathcal{C} \).
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Known: \( f(q) \in \mathcal{C} \) implies that, with \( g(q) := (f(0) - f(q))/q \), also \( g(q) \in \mathcal{C} \).

Taking \( f(q) = \rho^{(2)}(q) \), we have \( -\rho^{(1)}(q) \) is in \( \mathcal{C} \), and hence \( r'(\cdot) \) is negative, i.e., \( r(\cdot) \) is decreasing.

Similarly, \( \rho(q) \) is in \( \mathcal{C} \), and hence \( r(\cdot) \) is positive.
BUSY PERIOD: AN INTERMEZZO
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\[ \tau := \inf \{ t \geq 0 : Q_t = 0 \}, \text{ where } Q_0 \text{ has stationary distribution.} \]

\[ p(t) := \mathbb{P}(\tau > t). \]
BUSY PERIOD: AN INTERMEZZO

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\[ p(t) := \mathbb{P}(\tau > t). \]

Spectrally positive:

\[
\int_0^\infty e^{-\varrho t} p(t) dt = \int_0^\infty \left( \int_0^\infty e^{-\varrho t} \mathbb{P}(\tau(x) > t) dt \right) d\mathbb{P}(Q_0 < x) = \frac{1}{\varrho} \int_0^\infty \left( 1 - e^{-\psi(\varrho)q} \right) d\mathbb{P}(Q_0 < q).
\]

Application of ‘Pollaczek-Khinchine’:

\[
\int_0^\infty e^{-\varrho t} p(t) dt = \frac{1}{\varrho} - \varphi'(0) \frac{\psi(\varrho)}{\varrho^2}.
\]
BUSY PERIOD: AN INTERMEZZO

Spectrally negative:

Recall that \( \int_0^\infty e^{-qt} P(\tau > t) dt = q^{-1} (1 - \mathbb{E} e^{-q\tau}) \).

Known result:

\[
\mathbb{E} e^{-q\tau} = \int_0^\infty \beta_0 e^{-\beta_0 x} \mathbb{E} e^{-q\tau(x)} dx = \beta_0 \cdot \frac{\hat{\kappa}(q, \beta_0) - \hat{\kappa}(q, 0)}{\beta_0 \hat{\kappa}(q, \beta_0)};
\]

here \( \hat{\kappa}(q, \beta) \) relates to the transform of the so-called descending ladder process;

in spectrally-negative case: \( \hat{\kappa}(q, \beta) = (q - \Phi(\beta))/(\Psi(q) - \beta) \). Using that \( \Phi(\beta_0) = 0 \), we find

\[
\mathbb{E} e^{-q\tau} = \frac{\Psi(0)}{\Psi(q)},
\]

and in addition

\[
\int_0^\infty e^{-qt} p(t) dt = \frac{1}{q} \left( 1 - \frac{\Psi(0)}{\Psi(q)} \right).
\]
BUSY PERIOD: AN INTERMEZZO

Striking feature: transforms have the same *branching point* as the transforms of the correlation function!!
BUSY PERIOD: AN INTERMEZZO

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Spectrally positive, light tails ($\exists \alpha < 0 : \varphi(\alpha) = 0$): we roughly have

$$r(t) \sim p(t) \sim e^{\vartheta^* t},$$

where $\zeta$ is the minimizer of $\varphi(\cdot)$ and $\vartheta^* = \varphi(\zeta)$ the branching point of $\psi(\cdot)$. 
BUSY PERIOD: AN INTERMEZZO

Striking feature: transforms have the same *branching point* as the transforms of the correlation function!!

Spectrally positive, light tails ($\exists \alpha < 0 : \varphi(\alpha) = 0$): we roughly have

$$r(t) \sim p(t) \sim e^{\vartheta^* t},$$

where $\zeta$ is the minimizer of $\varphi(\cdot)$ and $\vartheta^* = \varphi(\zeta)$ the branching point of $\psi(\cdot)$.

Spectrally negative: we roughly have

$$r(t) \sim p(t) \sim e^{q^* t},$$

where $\zeta$ is the minimizer of $\Phi(\cdot)$ and $q^* = \Phi(\zeta)$ the branching point of $\Psi(\cdot)$. 
BUSY PERIOD: AN INTERMEZZO

Naïve simulation: estimate \( p(t) \) by

\[
S_n^{(NS)}(t) := \frac{1}{n} \sum_{i=1}^{n} 1\{\tau_i > t\}.
\]
Naïve simulation: estimate $p(t)$ by

$$S^{(NS)}_n(t) := \frac{1}{n} \sum_{i=1}^{n} 1\{\tau_i > t\}.$$ 

Number of runs needed to obtain estimate of given precision?

$$\sqrt{\frac{\text{Var} S^{(NS)}_n(t)}{p(t)}} < \varepsilon,$$

and realizing that

$$\text{Var} S^{(NS)}_n(t) = \frac{1}{n} p(t)(1 - p(t)) \approx \frac{p(t)}{n},$$

we see that the number of runs needed is roughly of order $1/p(t)$, i.e., exponentially increasing...
BUSY PERIOD: AN INTERMEZZO

A more clever algorithm can be constructed as follows (spectrally-positive case):

use Importance Sampling.

★ Let, in the interval \((0, t]\), the Lévy process be twisted with \(-\zeta = -\psi(\vartheta^*) > 0\).

Meaning: \(\varphi(\vartheta)\) replaced by \(\bar{\varphi}(\vartheta) := \varphi(\vartheta + \zeta) - \varphi(\zeta)\).

★ But what about distribution of \(Q_0\)?

Simulate \(Q_0\) from a \(\kappa\)-twisted version, i.e., a distribution with LT \(\mathbb{E}e^{-(\alpha-\kappa)Q_0}/\mathbb{E}e^{\kappa Q_0}\).

Call new measure \(Q_\kappa\).
BUSY PERIOD: AN INTERMEZZO

We simulate the process under $Q_\kappa$ till time $t$. Likelihood $L := L_A \cdot L_B$, where

- contribution due to the twisted Lévy process between 0 and $t$:
  $$L_A := e^{\psi(\vartheta^*)X_t} \cdot \mathbb{E}e^{-\psi(\vartheta^*)X_t} = e^{\psi(\vartheta^*)X_t} \cdot e^{\vartheta^*t}.$$  

- contribution due to the twisted queue at time 0 (use ‘Pollaczek-Khinchine’):
  $$L_B := e^{-\kappa Q_0} \cdot \mathbb{E}e^{\kappa Q_0} = e^{-\kappa Q_0} \cdot \frac{-\kappa \varphi'(0)}{\varphi(-\kappa)}.$$

Estimate $p(t)$ by, sampling under $Q_\kappa$,

$$S_n^{(IS)}(t) := \frac{1}{n} \sum_{i=1}^{n} L_i 1\{\tau_i > t\}.$$
BUSY PERIOD: AN INTERMEZZO

\[ L = e^{\psi(\vartheta^*)X_t} \cdot e^{\vartheta^*t} \cdot e^{-\kappa Q_0} \cdot \frac{-\kappa \varphi'(0)}{\varphi(-\kappa)}. \]

First option: not twisting \( Q_0 \) at all (i.e., choosing \( \kappa = 0 \)).

This does not work well: recalling that a necessary condition for \( \{\tau > t\} \) is \( \{Q_0 + X_t > 0\} \), we find

\[
\mathbb{E}_{Q_0} L^2 1\{\tau > t\} \leq \left( \frac{-\kappa \varphi'(0)}{\varphi(-\kappa)} \right)^2 e^{2\vartheta^*t} \mathbb{E}_{Q_0} e^{-2\kappa Q_0} e^{-2\psi(\vartheta^*)Q_0}. \tag{4}
\]

Logarithmic efficiency, meaning that the number of replications needed to obtain an estimate with a certain fixed precision grows subexponentially in the ‘rarity parameter’ \( t \):

\[
\limsup_{t \to \infty} t^{-1} \log \mathbb{E}_{Q_0} L^2 1\{\tau > t\} \leq 2 \vartheta^*.
\]

In other words: when picking \( \kappa = 0 \) we need to have \( \mathbb{E}_{Q_0} e^{-2\psi(\vartheta^*)Q_0} < \infty \) for logarithmic efficiency...

Not a priori clear....
BUSY PERIOD: AN INTERMEZZO

\[ L = e^{\psi(\vartheta^*)X_t \cdot \vartheta^*t} \cdot e^{-\kappa Q_0} \cdot \frac{-\kappa \varphi'(0)}{\varphi(-\kappa)}. \]

Second option: twisting with \( \kappa = -\zeta > 0. \)

Easy to see that we do get logarithmic efficiency here!
SIMULATION OF CORRELATION FUNCTION
But can we come up with an efficient simulation algorithm for $r(t)$?

Remember:

$$r(t) = \frac{\mathbb{E}Q_0Q_t - \mu^2}{\nu},$$

with $\mu := \mathbb{E}Q$ and $\nu := \text{Var} Q$ known...

We can estimate $\mathbb{E}Q_0Q_t - \mu^2$ by

$$T_n^{(NS)}(x) := \frac{1}{n} \sum_{i=1}^{n} Q_0^{(i)}Q_t^{(i)} - \mu^2.$$  

How many runs needed?
SIMULATION OF CORRELATION FUNCTION

Variance of this estimator:

$$\frac{1}{n} \cdot \text{Var}(Q_0Q_t) = \frac{\mathbb{E}(Q_0^2Q_t^2) - (\mathbb{E}(Q_0Q_t))^2}{n} \rightarrow \frac{(\mathbb{E}Q^2)^2 - (\mathbb{E}Q)^4}{n};$$

Conclude: number of runs needed roughly proportional to $1/r(t)^2$!!!
SIMULATION OF CORRELATION FUNCTION

Solution: coupling
We construct a coupling as follows.

Write:

\[ r(t) = \frac{1}{v} \cdot \mathbb{E}(Q_0 \cdot (Q_t - Q^*_t)), \]

where both \( Q \) and \( Q^* \) are stationary versions of the workload, and \( Q^*_t \) is independent of \( Q_0 \).

Construct this as follows: generate \( Q_0 \) and \( Q^*_0 \) independently, sampled from the stationary distribution of the workload. Now use exactly the same driving Lévy process \( X_t \) over \((0, t]\) to drive both \( Q_t \) and \( Q^*_t \) from their two independently generated initial conditions.

This makes \( Q_t \) and \( Q_0 \) correlated but \( Q^*_t \) and \( Q_0 \) independent.
We can estimate $\mathbb{E} Q_0 Q_t - \mu^2$ by

$$T_n^{(CS)}(x) := \frac{1}{n} \sum_{i=1}^{n} Q_0^{(i)} (Q_t^{(i)} - Q_t^{*(i)}).$$

What is performance of this estimator?
SIMULATION OF CORRELATION FUNCTION

Split $\mathbb{E}(Q_0 \cdot (Q_t - Q^*_t))$ into four terms, as follows.

Recall $M_t = \inf_{s \in (0,t]} X_s$. Then

$$ r(t) = r_{++}(t) + r_{+-}(t) + r_{-+}(t) + r_{--}(t), $$

where

$$
\begin{align*}
    r_{++}(t) & := \mathbb{E}(Q_0 \cdot (Q_t - Q^*_t) \cdot 1\{Q_0 + M_t > 0, Q^*_0 + M_t > 0\}), \\
    r_{+-}(t) & := \mathbb{E}(Q_0 \cdot (Q_t - Q^*_t) \cdot 1\{Q_0 + M_t > 0, Q^*_0 + M_t < 0\}), \\
    r_{-+}(t) & := \mathbb{E}(Q_0 \cdot (Q_t - Q^*_t) \cdot 1\{Q_0 + M_t < 0, Q^*_0 + M_t > 0\}), \\
    r_{--}(t) & := \mathbb{E}(Q_0 \cdot (Q_t - Q^*_t) \cdot 1\{Q_0 + M_t < 0, Q^*_0 + M_t < 0\}).
\end{align*}
$$
SIMULATION OF CORRELATION FUNCTION

Split $\mathbb{E}(Q_0 \cdot (Q_t - Q_t^*))$ into four terms, as follows.

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$$r_{--}(t) := \mathbb{E}(Q_0 \cdot (Q_t - Q_t^*) \cdot 1\{Q_0 + M_t < 0, Q_0^* + M_t < 0\}).$$

It is evident that $r_{--}(t) = 0$ as both queues have been empty (and this happens most of the time!)
SIMULATION OF CORRELATION FUNCTION

Key observation: \( |Q_t - Q^*_t| \leq |Q_0 - Q^*_0| \).

We therefore have:

\[
\text{Var} (Q_0(Q_t - Q^*_t)) \leq \mathbb{E}Q_0^2(Q_t - Q^*_t)^2 \leq \mathbb{E}Q_0^2(Q_0 - Q^*_0)^2.
\]

In addition:

\[
\mathbb{E}Q_0^2(Q_0 - Q^*_0)^2 \leq \mathbb{E}(Q_0^2(Q_0 - Q^*_0)^2 \cdot 1\{Q_0 + M_t > 0, Q^*_0 + M_t > 0\}) + \\
+ \mathbb{E}(Q_0^2(Q_0 - Q^*_0)^2 \cdot 1\{Q_0 + M_t > 0, Q^*_0 + M_t \leq 0\}) + \\
+ \mathbb{E}(Q_0^2(Q_0 - Q^*_0)^2 \cdot 1\{Q_0 + M_t \leq 0, Q^*_0 + M_t > 0\})
\]
SIMULATION OF CORRELATION FUNCTION

**Lemma:** in the spectrally-positive case

\[ \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(Q_0^k 1\{\tau > t\}) \leq \vartheta^* \]

(and \ldots \leq q^* in the spectrally-negative case).

Hence,

\[ \lim_{t \to \infty} \frac{1}{t} \log \text{Var}_0(Q_t(Q_t - Q_t^*)) \leq \vartheta^*. \]
SIMULATION OF CORRELATION FUNCTION

**Lemma:** in the spectrally-positive case

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(Q_0^k 1_{\{\tau > t\}}) \leq \vartheta^*
\]

(and \ldots \leq q^* in the spectrally-negative case).

Hence,

\[
\lim_{t \to \infty} \frac{1}{t} \log \text{Var} (Q_0(Q_t - Q_t^*)) \leq \vartheta^*.
\]

Consequently,

\[
\frac{\sqrt{\text{Var} T_n^{\text{(CS)}}(x)}}{r(t)} \approx \frac{\sqrt{e^{\vartheta^* t} / n}}{e^{\vartheta^* t}},
\]

so that number of runs needed grows roughly as \(1/r(t)\).

**Substantial improvement!**
SIMULATION OF CORRELATION FUNCTION

Augment coupling algorithm with Importance Sampling (as for busy period),
and we even get an algorithm for which the number of runs grows subexponentially!

This algorithm is called logarithmically efficient.
EXAMPLE I: REFLECTED BROWNIAN MOTION

Take $\mu = -1$, $\sigma^2 = 1$; remember

$$Q_t = X_t + \max \left\{ - \inf_{0 \leq s \leq t} X_s, Q_0 \right\}.$$  

$Q_0$ has an exponential distribution with mean $\frac{1}{2}$. 
EXAMPLE I: REFLECTED BROWNIAN MOTION

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$$Q_t = X_t + \max \left\{ -\inf_{0 \leq s \leq t} X_s, Q_0 \right\}.$$  

$Q_0$ has an exponential distribution with mean $\frac{1}{2}$.

Then we sample $X_t$ from a normal distribution with mean $-t$ and variance $t$; say it has value $z$. Using Brownian Bridge:

$$\mathbb{P} \left( -\inf_{0 \leq s \leq t} X_s \leq x \ \bigg| \ X_t = z \right) = \exp \left( -2 \frac{x}{t} (x + z) \right).$$  

Then it can be verified that

$$Y_z := \left( -\inf_{0 \leq s \leq t} X_s \ \bigg| \ X_t = z \right) \overset{d}{=} -\frac{z}{2} + \frac{1}{2} \sqrt{z^2 - 2t \log U},$$

where $U$ has a uniform distribution over (0, 1].

Hence: easy simulation of $Q_t$, requiring just three random numbers!
Perform $10^8$ runs per experiment;

the table gives the relative errors.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Naive</th>
<th>Coupling</th>
<th>IS</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$7.91 \cdot 10^{-4}$</td>
<td>35%</td>
<td>0.85%</td>
</tr>
<tr>
<td>12</td>
<td>$2.21 \cdot 10^{-4}$</td>
<td>75%</td>
<td>1.50%</td>
</tr>
<tr>
<td>14</td>
<td>$6.75 \cdot 10^{-5}$</td>
<td>133%</td>
<td>2.82%</td>
</tr>
<tr>
<td>16</td>
<td>$2.17 \cdot 10^{-5}$</td>
<td>151%</td>
<td>4.99%</td>
</tr>
<tr>
<td>18</td>
<td>$6.83 \cdot 10^{-6}$</td>
<td>160%</td>
<td>8.4%</td>
</tr>
<tr>
<td>20</td>
<td>$2.27 \cdot 10^{-6}$</td>
<td>188%</td>
<td>11.9%</td>
</tr>
</tbody>
</table>
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<th>IS</th>
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</tr>
</tbody>
</table>

Under importance sampling the relative error is more or less constant!
EXAMPLE II: M/M/1

We now take
\[ \varphi(\alpha) = \alpha - \lambda + \frac{\lambda \mu}{\mu + \alpha}, \]

It is readily checked that \( \zeta = -\mu + \sqrt{\lambda \mu}. \)
Perform $10^7$ runs per experiment;

the table gives the relative errors.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Naive</th>
<th>Coupling</th>
<th>IS</th>
</tr>
</thead>
<tbody>
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<td>7.0%</td>
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<td>$t = 60$</td>
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<td>41%</td>
<td>12.6%</td>
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<td>$t = 70$</td>
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<td>18.7%</td>
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<tr>
<td>$t = 80$</td>
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<td>31.8%</td>
</tr>
<tr>
<td>$t = 90$</td>
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<td>87%</td>
<td>46.4%</td>
</tr>
<tr>
<td>$t = 100$</td>
<td>$4.20 \cdot 10^{-5}$</td>
<td>101%</td>
<td>69.1%</td>
</tr>
</tbody>
</table>

Again: Under importance sampling the relative error is more or less constant!
OUTLOOK

What if we make it harder?

- Markov modulation? — then the processes are not necessarily ‘couplable’ at time 0.

  What to do if $Q_0$ and $Q_0^*$ correspond to different states of the modulating Markov chain?
  Wait till they meet? Force them to meet?

- more complex correlation structures? Fractional Brownian motion?
CONCLUSIONS

Analysis of correlation structure of process after imposing reflection map: is correlation structure inherited?

Hard to solve for the case of Gaussian inputs (such as fractional Brownian motion), but . . .

explicit analysis for the Lévy case: structural results, asymptotics, and efficient simulation.