We are concerned with a Kolmogorov operator in a separable Hilbert space $H$

$$N_0 \phi(x) := \frac{1}{2} \, \operatorname{Tr} [B B^* D_x^2 \phi(x)] + \langle Ax + b(x), D_x \phi(x) \rangle$$

where

- $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA},$
- $B \in L(H), \ B^*$ is the adjoint of $B,$
- $b : D(b) \subset H \to H$ is nonlinear.
If $H$ is infinite dimensional, $N_0$ can be considered as an elliptic operator (possibly degenerate) with infinitely many variables.

We want to study the parabolic equation

$$u_t(t, x) = (N_0 u)(t, x), \quad u(0, x) = u_0(x) \quad (1)$$

and the elliptic equation

$$\lambda \varphi - N_0 \varphi = f, \quad (2)$$

where $\lambda$ is a positive number and $u_0, f$ are given in suitable functional spaces which we shall discuss later.
The first goal would be to construct a general theory of second order elliptic and parabolic equations in Hilbert spaces, that is with infinitely many variables.

This project was essentially started by

Yu. Daleckij, Dokl. 66

and zz

L. Gross, JFA, 67
For further developments see the monographs,

Yu. Daleckij and S. V. Fomin, Kluwer, 91,

S. Cerrai, Springer 01,

Z. M. Ma and M. Röckner, Springer, 92

DP-Zabczyk, Cambridge 02.
One of the main motivation for studying the Kolmogorov equation (1) comes from the stochastic differential equation.

\[
\begin{aligned}
& dX = (AX + b(X))dt + BdW(t), \\
& X(0) = x \in H,
\end{aligned}
\]  

(3)

Several equations arising in the application are of this form. For instance: reaction-diffusion, Burgers or Navier–Stokes equation and many others.
Let us assume first that problem (3) has a unique solution \( X(t, x) \) (in a sense to be specified). Then there is a natural candidate for the solution to (1) given by

\[
u(t, x) = P_t \varphi(x), \quad x \in H, \ t \geq 0,
\]

where \( P_t \) is the transition semi-group

\[
P_t \varphi(x) = E[\varphi(X(t, x))], \quad \varphi \in B_b(H).
\]
There are no chances, however, to check in general, even for a very regular $\varphi$, that $u(t, x) = \mathbb{E}[\varphi(X(t, x))]$ is a solution of (1) because $X(t, x)$ is not differentiable enough.

Also it is not a priori clear for which $\varphi$ the expression of $N_0 \varphi$ is meaningful, because $A$ and $b$ are not everywhere defined.

So, one of our main concern will be the choice of a suitable space of test functions $\mathcal{D}$ such that $N_0 \varphi$ is meaningful for each $\varphi \in \mathcal{D}$.  

Kolmogorov equations in Hilbert spaces 1
We shall see often that $P_t$ acts on $C_b(H)$, which means that it is Feller.

However, $P_t$ it is not a strongly continuous semi group in $C_b(H)$ in general, because for $\varphi \in C_b(H)$ the convergence

$$\lim_{t \to 0} P_t\varphi(x) = \varphi(x)$$

only holds point-wise.

We shall say that $P_t$ is a $\pi$-semigroup.
We can generalize, however, the usual definition of infinitesimal generator $N$ of $P_t$, so that the resolvent of $N$ coincides with the Laplace transform of $P_t$,

$$ (\lambda - N)^{-1} f(x) = \int_0^{+\infty} e^{-\lambda t} P_t f(x) dt. \quad (4) $$

Since $P_t$ is a contraction semigroup in $C_b(H)$, $L$ is $m$-dissipative, so that its resolvent set includes the half-line $(0, +\infty)$.

Notice that the integral in (4) is defined and convergent only pointwise, that is for each $x \in H$. 

Kolmogorov equations in Hilbert spaces 1
Now the problem arises to see the relationship between the abstract operator $N$ and the Kolmogorov operator $N_0$.

We expect that the space of test function $\mathcal{D}$ is chosen in such a way then $N_0\varphi$ coincides with $N\varphi$ for all $\varphi \in \mathcal{D}$.

We shall also wish that the restriction of $N$ to $\mathcal{D}$ determines $N$, in other words that $\mathcal{D}$ is a core for $N$. 
We also notice that if the space of test functions $\mathcal{D}$ is a core then we can exploit the explicit properties of the differential operator $N_0$.

Let us give an example, assuming that $\mathcal{D}$ is an algebra of regular functions.

Then, given $\varphi \in \mathcal{D}$ we see, by an elementary computation, that

$$N_0(\varphi^2) = 2N_0\varphi\varphi + |B^*D\varphi|^2, \quad \forall \varphi \in \mathcal{D}. \quad (5)$$
Assume now that $\mu$ is an infinitesimally invariant probability measure for $N_0$,

$$\int_H N_0 \varphi \, d\mu = 0, \quad \forall \varphi \in \mathcal{D}$$  \hspace{1cm} (6)

Then by (5) we deduce the important Identité du carré du champs

$$\int_H N \varphi \varphi \, d\mu = -\frac{1}{2} \int_H |B^* D\varphi|^2 \, d\mu, \quad \forall \varphi \in D(N).$$
Another problem which arises now is to study the regularity of the strong solution $\varphi$ of (6) following that of $f$

and also (this is more difficult) to try to characterize the domain of $N$.

We can follow this program only in particular cases, which we shall discuss later.
We shall also consider another approach, namely to look for a realization of $P_t$ in $L^p(H, \mu)$, $p \geq 1$, where $\mu$ is an invariant measure for $P_t$, that is
\[
\int_H P_t \varphi \, d\mu = \int_H \varphi \, d\mu, \quad \forall \varphi \in C_b(H).
\]
Then one can uniquely extend $P_t$ to $L^p(H, \mu)$ for all $p \geq 1$ as a strongly continuous semigroup of contractions, which we still denote by $P_t$, whereas we call $N_p$ its infinitesimal generator.
Again the problem arises to find a core of nice functions for $N_p$. As a core we shall often choose the space $\mathcal{E}_A(H)$ of all linear combinations of real parts of exponential functions

$$\varphi_h(x) = e^{i\langle x, h \rangle}, \quad h \in D(A^*), \; x \in H.$$ 

Notice that,

$$N_0\varphi_h(x) = \left(-\frac{1}{2} |B^* h|^2 + i\langle x, A^* h \rangle + i\langle b(x), h \rangle\right) \varphi_h(x). \quad (7)$$

So, we must assume at least that $\int_H |b(x)|\mu(dx) < +\infty$. 

Kolmogorov equations in Hilbert spaces 1
We notice that proving that $\mathcal{E}_A(H)$ is a core for $N_p$ is equivalent to show that the linear operator

$$
N_0 \varphi(x) = \frac{1}{2} \text{Tr} [BB^* D_x^2 \varphi(x)] + \langle x, A^* D_x \varphi(x) \rangle \\
+ \langle b(x), D_x \varphi(x) \rangle, \quad \varphi \in \mathcal{E}_A(H),
$$

has a unique $m$–dissipative extension in $L^p(H, \mu)$.

We shall prove essential $m$–dissipativity of $N_0$ for some concrete example.
The case when problem (3) is ill-posed

There are interesting situations where existence and uniqueness for the stochastic differential equation (3) are not known.

In this case the transition semigroup $P_t$ is not given in advance.

Then one can try to study directly the Kolmogorov equation

$$\lambda \varphi(x) - \frac{1}{2} \operatorname{Tr} [BB^* D_x^2 \varphi(x)] - \langle Ax + b(x), D_x \varphi(x) \rangle = f.$$  \hspace{1cm} (8)
One possibility is to introduce and solve an approximating equation

\[ \lambda \varphi_n(x) - \frac{1}{2} \text{Tr} [BB^* D_x^2 \varphi_n(x)] - \langle Ax + b_n(x), D_x \varphi_n(x) \rangle = f, \] (9)

then to find suitable estimates for \( \varphi_n \) and their derivatives, independent of \( n \), and show the convergence of a subsequence of \( (\varphi_n) \) to a solution, in a suitable sense, of (8).

This method was used in DP–A. Debussche JMPA 03, to prove existence, but not uniqueness, of a solution of (8) in the case of 3D-NS.
A second possibility is to show existence of an infinitesimally invariant measure $\mu$ of $N_0$ and then to prove the essential $m$-dissipativity of $N_0$ in $L^2(H, \mu)$.

This method was used in DP–Röckner PTRF 02, to study some equations with dissipative and singular coefficients.

If one is able to prove essential $m$-dissipativity of $N_0$, then one can try to construct a unique martingale solution of (3).
Finally, an important approach to be mentioned is the one of **Dirichlet forms** in infinite dimensional spaces started in


S. Albeverio and M. Röckner, PTRF 91.

See also the monograph

Z. M. Ma and M. Röckner, Springer-Verlag, 92

and

W. Stannat, Memoirs AMS, 99.
1) Ornstein–Uhlenbeck operators.

2) Bounded perturbations of O.U.

3) Dissipative perturbations of O.U.

4) Fokker–Planck equations with singular coefficients.

5) Porous media equations.
Here we consider the Kolmogorov operator in $H$ of the form

$$L_0 \varphi(x) := \frac{1}{2} \text{Tr} \left[ BB^* D_x^2 \varphi(x) \right] + \langle x, A^* D_x \varphi(x) \rangle$$

where

- $A : D(A) \subset H \rightarrow H$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA}$,
- $B \in L(H)$, $B^*$ is the adjoint of $B$,
- The linear operator

$$Q_t = \int_0^t e^{sA} BB^* e^{sA^*} ds, \quad t \geq 0,$$  \hspace{1cm} (10)

is of trace class for any $t > 0$. 

1 Ornstein–Uhlenbech operators
Condition (10) is necessary. In fact, setting $u(t, x) = v(t, e^{tA}x)$ we transform the problem

$$u_t(t, x) = (L_0 u)(t, x), \quad u(0, x) = u_0(x), \quad (11)$$

in

$$v_t(t, x) = \frac{1}{2} \text{Tr} \left[ e^{tA}BB^* e^{tA^*} D_x^2 v(t, x) \right].$$

This can be easily solved

$$v(t, x) = \int_H u_0(y) N_{Q_t}(dy)$$

where $Q_t$ is given by (10).
The stochastic differential equation corresponding to $L_0$ is the following

$$\begin{cases} \quad dX = AXdt + BdW(t), \\ \quad X(0) = x \in H, \end{cases}$$  \hspace{1cm} (12)$$

which can be solved by the variation of constants formula

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}dW(s) =: e^{tA}x + W_A(t).$$

$W_A(t)$ is called the stochastic convolution.

Kolmogorov equations in Hilbert spaces 1
The stochastic convolution

We are given

- \((e_k)_{k \in \mathbb{N}}\) complete orthonormal system in \(H\).
- \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) filtered probability space.
- \((W_k)_{k \in \mathbb{N}}\) sequence of real independent standard Brownian motions in \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\).
The we define

\[ W_A(t) : = \int_0^t e^{(t-s)A} B dW(s) \]

[13]

\[ = \sum_{k=1}^{\infty} \int_0^t e^{(t-s)A} B e_k dW_k(s), \quad t \geq 0. \]
Since

\[ \mathbb{E} |W_A(t)|^2 = \sum_{k=1}^{\infty} \int_0^t |e^{(t-s)A} B e_k|^2 ds \]

\[ = \int_0^t \text{Tr} \left[ e^{sA} B B^* e^{sA^*} \right] ds \]

the series in (13) is convergent.
For each \( x \in H \) we consider the process

\[
X(t, x) := e^{tA}x + W_A(t), \quad t \geq 0.
\]

\( X(t, x) \) is a Gaussian random variable \( N_{e^{tA}x, Q_t} \),

where

\[
Q_t = \int_0^t e^{sA}BB^* e^{sA^*} \, ds, \quad t \geq 0,
\]

Kolmogorov equations in Hilbert spaces 1
The corresponding transition semigroup $R_t$ is

$$R_t \varphi(x) : = \mathbb{E}[\varphi(X(t, x))]$$

$$= \int_H \varphi(y) N_{e^{tA}x, Q_t}(dy)$$

$$= \int_H \varphi(e^{tA}x + y) N_{Q_t}(dy), \quad \varphi \in B_{b,n}(H).$$
For any $n \in \mathbb{N}$, $B_{b,n}(H)$ is the Banach space of all Borel functions $\varphi : H \to \mathbb{R}$ such that

$$\|\varphi\|_{0,n} := \sup_{x \in H} \frac{|\varphi(x)|}{1 + |x|^n} < +\infty.$$ 

$C_{b,n}(H)$ is the closed subspace of $B_{b,n}(H)$ of all functions $\varphi$ such that $\frac{|\varphi(\cdot)|}{1 + |\cdot|^n}$ is uniformly continuous.

It is easy to see that the semigroup $R_t$ acts both on $B_{b,n}(H)$ and in $C_{b,n}(H)$.

If $n = 0$ we write $C_{b,n}(H) = C_b(H)$ and $\|\varphi\|_{0,n} = \|\varphi\|_0$. 

Kolmogorov equations in Hilbert spaces 1
We first consider $R_t$ acting in $C_b(H)$. In this case we have

$$\|R_t\varphi\|_0 \leq \|\varphi\|_0.$$ 

If $A \neq 0$, $R_t$ is not a strongly continuous semigroup.

In fact if $\varphi_h(x) = e^{i\langle x, h \rangle}$ the limit

$$\lim_{t \to 0} R_t \varphi_h(x) = \lim_{t \to 0} e^{-\frac{1}{2}\langle Q_t h, h \rangle} e^{i\langle e^{tA}x, h \rangle} = \varphi_h(x), \quad x \in H,$$

is not uniform in $x$ for any $h \neq 0$ (unless $A = 0$).
What we can say is that for any $\varphi \in C_b(H)$ one has

$$\lim_{t \to 0} R_t \varphi(x) = \varphi(x), \quad \forall \ x \in H$$

and

$$\sup_{t \geq 0} \| R_t \varphi \|_0 \leq \| \varphi \|_0.$$
Let $f \in C_b(H)$ and $\{f_n\} \subset C_b(H)$. We say that $\{f_n\}$ is $\pi$-convergent to $f$ and write $f_n \xrightarrow{\pi} f$ if

1. $\lim_{n \to \infty} f_n(x) = f(x), \quad \forall \ x \in H$

2. $\sup_{n \in \mathbb{N}} \|f_n\|_0 < \infty$. 
π-convergence was called bp-convergence in


and


see also

B. Goldys and M. Kocan, JDE 2001, were a suitable topology corresponding to π-convergence was constructed.
We define the infinitesimal generator $L$ of $R_t$ in $C_b(H)$ as follows. The domain $D(L)$ of $L$ is the set of all $\varphi \in C_b(H)$ such that

- There exists

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} (R_\epsilon \varphi(x) - \varphi(x)) =: L\varphi(x)$$

for all $x \in H$.

- $\sup_{\epsilon \in (0,1]} \frac{1}{\epsilon} \| R_\epsilon \varphi - \varphi \|_0 < \infty$. 
We are going to study the resolvent set $\rho(L)$ and the resolvent $R(\lambda, L) = (\lambda - L)^{-1}$ of $L$.

Results and proofs are straightforward generalizations of the classical Hille–Yosida theorem (the difference being that $R_t$ is not strongly continuous), and so they will be only sketched.

For details see E. Priola, Studia Math., 1999.
Proposition 1

Let $L$ be the infinitesimal generator of $R_t$ in $C_b(H)$. Then

(i) $(0, +\infty) \subset \rho(L)$ and we have

$$R(\lambda, L)f(x) = \int_0^{+\infty} e^{-\lambda t} R_t f(x) dt, \quad f \in C_b(H), \, \lambda > 0, \, x \in H.$$ 

(Note that the integral below is only pointwise defined.)

Moreover,

$$\| R(\lambda, L)f \|_0 \leq \frac{1}{\lambda} \| f \|_0, \quad \lambda > 0, \, f \in C_b(H).$$

(ii) If $f \in C_b(H)$ and $\{f_n\} \subset C_b(H)$ is a sequence such that $f_n \xrightarrow{\pi} f$, we have $R(\lambda, L)f_n \xrightarrow{\pi} R(\lambda, L)f$. 
Let $f \in C_b(H)$. Write for any $\lambda > 0$ and any $x \in H$

$$F(\lambda)f(x) = \int_0^{+\infty} e^{-\lambda t} R_t f(x) dt.$$ 

It is easy to check that $F(\lambda)f \in C_b(H)$, that

$$\frac{d}{dh} R_h F(\lambda)f(x)|_{h=0} = \lambda \int_0^{+\infty} e^{-\lambda s} R_s f(x) ds - f(x)$$

$$= \lambda F(\lambda)f(x) - f(x),$$
and that the family of linear operators \((\Delta_h)_{h \in (0,1]}\)

\[
\Delta_h F(\lambda)f := \frac{1}{h} (R_h F(\lambda)f - F(\lambda)f)
\]

is equi-bounded in sup norm. This implies that \(F(\lambda)f \in D(L)\) and \((\lambda - L)F(\lambda)f = f\)

Analogously one sees that if \(\varphi \in D(L)\) we have \(F(\lambda)(\lambda - L)\varphi = \varphi\).

This will achieve the proof that \(\rho(L) \supset (0, +\infty)\).

Finally, (ii) is straightforward. \(\square\)
Remark

One can make similar considerations by replacing space $C_b(H)$ with $C_{b,n}(H)$.

When some confusion may arise we will denote by

$$D(L, C_{b,n}(H))$$

the domain of the generator $L$ of semigroup $R_t$ acting in $C_{b,n}(H)$.

For details see DP, Birkhäuser, 2004.
Identification of $L$

It seems to be difficult to characterize the domain of $L$ in $C_b(H)$.

So, we shall try to find a subset $\mathcal{D}$ of $C_b(H)$ with the following properties:

- $L$ looks like an explicit differential operator on $\mathcal{D}$.
- $\mathcal{D}$ is stable for $R_t$.
- For any $\varphi \in D(L)$ there is a two-indices sequence $(\varphi_{n_1,n_2}) \subset \mathcal{D}$, $\pi$-convergent to $\varphi$ and such that $(L\varphi_{n_1,n_2})$ is $\pi$-convergent to $L\varphi$.

We shall call such a set $\mathcal{Y}$ a core of $L$. 

Kolmogorov equations in Hilbert spaces 1
Exponential functions

We denote by $\mathcal{E}_A(H)$ the linear span of the real parts of functions

$$\varphi_h(x) := e^{i\langle x, h \rangle}, \quad x \in H$$

with $h \in D(A^*)$.

Recalling the Fourier transform formula of a Gaussian measure we see that

$$R_t \varphi_h(x) = e^{i\langle x, e^{tA^*}h \rangle} e^{-\frac{1}{2}\langle Q_t h, h \rangle}, \quad h \in H.$$  

Therefore $\mathcal{E}_A(H)$ is stable for $R_t$.  

Kolmogorov equations in Hilbert spaces 1
Proposition 2

Let $\varphi \in C_b(H)$, $k = 0, 1$. Then there exists a double sequence $(\varphi_{m,n}) \subset \mathcal{E}_A(H)$, $\pi$-convergent to $\varphi$ in $C_b(H)$.

Proof. The result is well known in finite dimensions (with an one index sequence).

In infinite dimensions we approach $\varphi(x)$ by $\varphi(P_n x)$ where $(P_n)$ is a sequence of finite dimensional projectors convergent to the identity. □
Now we can easily see that

$$\mathcal{E}_A(H) \subset D(L, C_{b,1}(H))$$

and

$$L\varphi = \frac{1}{2} \text{Tr} [BB^* D^2 \varphi] + \langle x, A^* D\varphi \rangle, \quad \forall \varphi \in D(L, C_{b,1}(H)). \quad (15)$$
Proposition 3

$\mathcal{E}_A(H)$ is a core for $(L, D(L, C_{b,1}(H)))$.

Proof. $\mathcal{E}_A(H)$ is stable for $R_t$ and $\pi$-dense in $C_{b,1}(H)$. The proof is an arrangement of a classical result on $C_0$-semigroups.

Obviously

\[ E_A(H) \notin D(L, C_b(H)). \]

One can see, however, that integrals of exponential functions belong to $D(L, C_b(H))$. More precisely, let $I_A(H)$ be the linear span of real parts of all functions of the form

\[ \int_0^a g(s) e^{i \langle e^{sA}x, h \rangle} \, ds : a > 0, \ h \in D(A^*), \ g \in C^1([0, a])). \]

then one can check that $I_A(H)$ is a core for $D(L, C_b(H))$. 

Kolmogorov equations in Hilbert spaces 1
Notice only that if

\[ \varphi(x) = \int_0^a e^{i\langle e^{sA}x, h \rangle} ds, \]

we have

\[ \langle D\varphi(x), z \rangle = i \int_0^a e^{i\langle e^{sA}x, h \rangle} \langle e^{sA}z, h \rangle ds, \]

so that

\[ \langle D\varphi(x), Ax \rangle = \int_0^a e^{i\langle e^{sA}x, h \rangle} ds - \int_0^a e^{i\langle x, h \rangle} ds. \]
Hypoellipticity of $L$

Here we consider the operator $L$ acting in $C_b(H)$. We call $L$ hypoelliptic if

$$\varphi \in B_b(H), \ t > 0 \Rightarrow R_t\varphi \in C_b^\infty(H).$$

**Theorem 4**

$L$ is hypoelliptic if and only if

$$\exp^{tA}(H) \subset Q_t^{1/2}(H), \ \forall \ t > 0.$$  \hspace{1cm} (16)
Condition (16) is equivalent to null controllability (in any times) of the following deterministic system

\[ X'(t) = AX(t) + Bu(t), \quad X(0) = x. \]  

(17)

System (17) is called null controllable in time \( T > 0 \) if for any \( x \in H \) there exists \( u \in L^2(0, T; H) \) such that \( X(T) = 0 \).
Moreover the minimal needed energy for driving $X$ to 0 in time $T$ is precisely given by $|\Lambda_T x|^2$, where

$$\Lambda_T := Q_T^{-1/2} e^{TA}.$$  \hspace{1cm} (18)

See J. Zabczyk Institute of Mathematics, Polish Academy of Sciences, 1981

and

DP-J. Zabczyk, Cambridge 02.
The case when $L$ is elliptic

We say that $L$ is elliptic if $(BB^*)^{-1} \in L(H)$.

In this case $L$ is hypoelliptic, that is condition (16) is fulfilled. Take in fact for simplicity $B = I$ and set

$$u(t) = -\frac{X}{T} e^{tA}, \quad t \in [0, T].$$

Then

$$X(T) = e^{TA}x - \frac{1}{T} \int_0^T e^{(T-s)A} e^{sA} ds = 0.$$

So, system (17) is null controllable and

$$\|\Lambda_t\| \leq \frac{1}{\sqrt{t}}, \quad \forall \ t > 0.$$
Proof of Theorem 4

We shall only sketch the proof of the “if” part, for details see DP–J.Zabczyk, Cambridge, 02.

So, we assume that

$$e^{tA}(H) \subset Q_t^{1/2}(H)$$

and take $\varphi \in B_b(H)$. In view of the Cameron–Martin theorem we have

$$N_{e^{tA}x, Q_t} \ll N_{Q_t}$$
Setting now
\[ \rho(x, y) : \frac{dN_{e^{tA}x, Q_t}}{dN_{Q_t}}(y) = e^{-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_{t}^{-1/2}y \rangle}, \]
we have
\[ R_t \varphi(x) = \int_{H} e^{-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_{t}^{-1/2}y \rangle} \varphi(y) N_{Q_t}(dy). \]
Now the conclusion follows by straightforward computations. □
We are here concerned with the elliptic Kolmogorov equation

$$\lambda \varphi - L\varphi = f, \quad (19)$$

where \( \lambda > 0 \) and \( f \in C_b(H) \).

Since the resolvent set of \( L \) includes \((0, +\infty)\), equation (19) has a unique solution \( \varphi \in D(L) \).

We are going to study the regularity of \( \varphi \).
Regularity results

We need the following new assumption.

**Hypothesis**

There exists $\alpha \in [1/2, 1)$ and $c_\alpha > 0$ such that

$$\|\Lambda_t\| \leq c_\alpha t^{-\alpha}, \quad \forall \ t \geq 0.$$  \hfill (20)

We have seen that this assumption is fulfilled when $L$ is elliptic, that is when $(BB^*)^{-1} \in L(H)$, with $\alpha = \frac{1}{2}$. 

Kolmogorov equations in Hilbert spaces
Proposition 5

Assume that (20) is fulfilled.

Let $\lambda > 0$, $f \in C_b(H)$ and let $\varphi$ be the solution of equation (19). Then $\varphi \in C^1_b(H)$ and there exists $K_\alpha > 0$ such that

\[ |D\varphi(x)| \leq \frac{K_\alpha}{\lambda^{\alpha-1}} \|\varphi\|_0, \quad \forall \ x \in H. \quad (21) \]

Consequently $D(L) \subset C^1_b(H)$. 
Proof

It follows by taking the Laplace transform in the formulae

\[
\langle DR_t \varphi(x), h \rangle = \int_H \langle \Lambda_t h, Q_t^{-1/2} y \rangle \varphi(e^{tA} x + y) N_Q(t \, dy),
\]

\[
|DR_t \varphi(x)| \leq \|\Lambda_t\| \|\varphi\|_0, \quad \forall t > 0.
\]

\[\square\]
The elliptic equation can be studied in $f \in C_{b,1}(H)$ which minor modifications.
Let us notice, for further use the following result.

### Proposition 6

Assume that hypotheses (16) and (20) hold.
For each $\varphi \in D(L, C_{1,b}(H))$ there exists a two indices sequence $(\varphi_{m,n}) \in \mathcal{S}_A(H)$ such that

1. \[ \lim_{m \to \infty} \lim_{n \to \infty} \varphi_{m,n}(x) = \varphi(x), \quad \lim_{m \to \infty} \lim_{n \to \infty} L\varphi_{m,n}(x) = L\varphi(x) \quad \text{and} \]
   \[ \lim_{m \to \infty} \lim_{n \to \infty} D\varphi_{m,n}(x) = D\varphi(x) \quad \text{for all } x \in H. \]

2. \[ |\varphi_{m,n}(x)| + |L\varphi_{m,n}(x)| + |D\varphi_{m,n}(x)| \leq \|\varphi\|_{b,1}(1 + |x|), \quad \text{for all } x \in H. \]