Kinetic equations from stochastic dynamics in continuum

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Non-linear PDE and SPDE

as phenomenological models of complex systems.

Reaction-diffusion equations (RDE)

\[
\frac{\partial u}{\partial t} = \Delta u + f(u), \quad u = u(t, x)
\]

in combustion theory, bacterial growth, nerve propagation, epidemiology, genetics etc.

RDE = Allen-Cahn = Ginzburg-Landau

Fisher equation: \( f(s) = s(1 - s) \).
We observe an active recent development in the study of non-local versions of RDE:

\[
\frac{\partial u}{\partial t} = J \ast u - u + f(u), \quad u = u(t, x).
\]

\[
u(0, x) = \varphi(x), \quad x \in \Omega \subseteq \mathbb{R}^d
\]

\[
0 \leq J \in L^1, \quad \|J\|_1 = 1 \text{ jump kernel.}
\]

Just few references:
Coville, Dupaigne (2008)
Ignat, Rossi (2010)
Berestycki, Nadin, Ryzhik (2009)
Pan, Li, Lin (2009)
Zhang, Li, Sun (2010)
Actually, such kind of equation was introduced by Kolmogorov, Petrovsky and Piskunov (KPP) in 1937 (!) as a way to derive Fisher equation.

Local RDE (macroscopic description) are approximations to non-local RDE (mesoscopic description)
**Problem**: derivation of meso- and macroscopic equations (deterministic or stochastic) from microscopic models.

There are known different techniques:
- scaling limits for dynamics (hydrodynamic, Vlasov, Landau etc.)
- scaling of fluctuations (equilibrium or non-equilibrium)
- closure of (infinite linear) moment systems
- hierarchical chains (BBGKY etc.)
- particle approximation method
Three levels in physics

In context of the theory of classical gases:

\( (Mi) \) is the level of particle dynamics (Newtons laws)

\( (Me) \) is the level of Boltzmann description

\( (Ma) \) is the level of continuum description.
Microscopic Stochastic Systems

In mathematical terms we are interested in the links between the following mathematical structures:

\( (M_1) \) the micro–scale of stochastically interacting entities (cells, individuals, . . . ), in terms of (linear) Markov semigroups (or corresponding processes)

\( (M_E) \) the meso–scale of statistical entities, in terms of continuous nonlinear semigroups related to the solutions of nonlinear Boltzmann–type non-local kinetic equations

\( (M_A) \) the macro–scale of densities of interacting entities (in terms of dynamical systems related to reaction–diffusion type equations).
Configuration space:

\[ \Gamma := \{ \gamma \subset \mathbb{R}^d \mid |\gamma \cap \Lambda| < \infty \text{ for any compact } \Lambda \subset \mathbb{R}^d \} . \]

| · | = cardinality of the set.
\[ \gamma = \{ x_k \mid k \in \mathbb{N} \} \text{ with different points} \]
\[ \Gamma \text{ is a Polish space.} \]

n-point configuration space:

\[ \Gamma^{(n)} := \{ \eta \subset \mathbb{R}^d \mid |\eta| = n \} , \quad n \in \mathbb{N}_0 . \]

Space of finite configurations:

\[ \Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)} . \]
Dynamics of configurations

Deterministic dynamics:
- Hamiltonian dynamics
- Interacting particle dynamical systems

Vlasov equation
due to Braun/Hepp, 1977 and Dobrushin, 1979:
we study asymptotic for $N \to \infty$ of the solution to

$$\frac{dx_i(t)}{dt} = A(x_i(t)) + N^{-1} \sum_{j=1}^{N} B(x_i(t) - x_j(t)),$$

$x_i \in \mathbb{R}^d$, $i = 1, \ldots, N$.

Actually, shall be $x_i^{(N)}$ notation.
Empirical measure:

\[ \mu^N_t = \frac{1}{N} \sum_i \delta_{x_i(t)} \]

We assume (initial density approx.)

\[ \mu^N_0(dx) \to \rho_0(x)dx, \quad N \to \infty. \]

VE for the limiting density (sic!):

\[ \frac{\partial \rho_t(x)}{\partial t} = -Tr(\nabla_x (A(x)\rho_t(x))) - Tr \nabla_x \left\{ \rho_t(x) \int_{\mathbb{R}^d} B(x - y) \rho_t(y) dy \right\}. \]

Finite volume assumption?
We are dealing with \( L^1 \)-solutions only!
No results at all for bdd densities = infinite systems.
Method is based on the use of the equation of motion and effectively means a version of LLN

**VE** for Hamiltonian dynamics from BBGKY hierarchy (heuristic derivation):
H.Spohn, 1980

V.P.Maslov, 1982: derivation of Vlasov equation from BBGKY for ”random number of particles” (=zero density system).

Rigorous derivation for positive densities meets problems:
the lack of detailed knowledge about the solution to BBGKY.

**VE** appeared originally in plasma physics
and in the stellar dynamical problems.
Markov evolutions in continuum:

- Diffusions (e.g., gradient diffusion)
- Jumping particles Markov processes (e.g., Kawasaki dynamics)
- Birth-and-death stochastic dynamics (e.g., Glauber, IBM in spatial ecology)
- Other stochastic IPS in $\mathbb{R}^d$

Questions:

- What is possible concept of related Vlasov equations?
- May we work with infinite particle dynamics?
- Is there a notion of a limiting IPS dynamics which creates Vlasov equation?
- Other scalings and other kinetic equations?
Lattice framework: kinetic equations from lattice stochastic dynamics

For lattice stochastic dynamics and their hydrodynamic scalings

we have several excellent results by

Lebowitz, Presutti, Varadhan, Yau, Kipnis, Landim

and many other authors

showing how macroscopic kinetic equations appear from microscopic stochastic processes.

Particular examples of non-local RDE were derived from lattice stochastic dynamics by Durrett, 1995.

Front propagation in these mesoscopic models analyzed by Perthame/Souganidis, 2005.

Scaling of lattice stochastic dynamics to super-processes etc.
Our approach is based on the study of state evolution for the considered systems.

**Markov dynamics** of IPS ⇒ **evolution of states** (measures)

Particular scalings ⇒ different kinds of kinetic equations

General discussion for **Hamiltonian dynamics**:
[Dobrushin/Sinai/Suhov], 1985.

Interacting diffusions:
**McKean-Vlasov limit**
which is a **stochastic version**
of the deterministic case.
Classical results by McKean, Dawson, Gärtner, Shiga et. al.
Markov evolutions

Let $L$ be a Markov pre-generator defined on some set of functions $\mathcal{F}(\Gamma)$ given on the configuration space $\Gamma$.

(Backward) Kolmogorov equation:

$$\frac{\partial F_t}{\partial t} = LF_t,$$

$$F_t|_{t=0} = F_0;$$

Duality: $< F, \mu > := \int_{\Gamma} F d\mu$

(Forward) Kolmogorov = Fokker-Planck equation:

$$\frac{\partial \mu_t}{\partial t} = L^* \mu_t,$$

$$\mu_t|_{t=0} = \mu_0.$$
Vlasov scaling

Initial distribution: $\mu_0 \in M^1_{fm}(\Gamma)$ with correlation function $k_0 : \Gamma_0 \rightarrow \mathbb{R}_+$, $k_0 = \{k_0^{(n)}, n \geq 0\}$.

$\mu_t \in M^1(\Gamma)$ the distribution at time $t > 0$ and $k_t$ its correlation function.

\[
\begin{align*}
\frac{d\mu_t}{dt} &= L^* \mu_t \\
\mu_t \bigg|_{t=0} &= \mu_0,
\end{align*}
\]

where $L^*$ is the adjoint to the generator on functions

\[
\begin{align*}
\frac{dF_t}{dt} &= LF_t \\
F_t \bigg|_{t=0} &= F_0.
\end{align*}
\]

\[
\begin{align*}
\frac{dk_t}{dt} &= L^\Delta k_t \\
k_t \bigg|_{t=0} &= k_0
\end{align*}
\]

where $L^\Delta := \hat{L}^*$ is the generator of a semigroup $T_t^\Delta := \hat{T}_t^*$. 
1st step in VS

Choose the initial state of the system:

∀ε > 0 correlation functions $k_0^{(ε)}$ as $ε → 0$:

$$k_{0, \text{ren}}^{(ε)}(η) := ε|η|k_0^{(ε)}(η) → r_0(η), \quad ε → 0, \quad η ∈ Γ_0,$$

where correlation function $r_0$ will be chosen properly.

In the case of

$$r_0(η) = e^λ(ρ_0, η) = \prod_{x ∈ η} ρ_0(x), \quad η ∈ Γ_0$$

the assumption about the initial conditions leads to a Poisson measure:

$$μ_0^{(ε)} → πρ_0,$$

where $μ_0^{(ε)}$ has correlation function $ε|η|k_0^{(ε)}(η)$. 
Scaling of the generator:

\[ L \mapsto L_\varepsilon. \]

The concrete type of this scaling will depend on the model.

Suppose that there exist solution of the correlation functional evolution

\[
\begin{cases}
    \frac{dk_t^{(\varepsilon)}}{dt} = L_\varepsilon k_t^{(\varepsilon)} \\
    k_t^{(\varepsilon)} \big|_{t=0} = k_0^{(\varepsilon)}
\end{cases}
\]

We expect (and this will be shown in concrete models) that order of the singularity in \( \varepsilon \) for this solution will be the same as for initial function \( k_0^{(\varepsilon)} \).
3rd step in VS

We consider

\[ k_{t, \text{ren}}^{(\varepsilon)}(\eta) := \varepsilon |\eta| k_{t}^{(\varepsilon)}(\eta), \quad \eta \in \Gamma_0, \]

and want to show that

\[ k_{t, \text{ren}}^{(\varepsilon)}(\eta) \to r_t(\eta), \quad \varepsilon \to 0, \ \eta \in \Gamma_0. \]

In fact, we consider renormalized version of the evolution equation

\[
\begin{cases}
\frac{dk_{t, \text{ren}}^{(\varepsilon)}}{dt} = L^{\Delta}_{\varepsilon, \text{ren}} k_{t, \text{ren}}^{(\varepsilon)} \\
k_{t, \text{ren}}^{(\varepsilon)}|_{t=0} = k_{0, \text{ren}}^{(\varepsilon)}
\end{cases}
\]

where

\[ L^{\Delta}_{\varepsilon, \text{ren}} = \varepsilon |\eta| L^{\Delta}_\varepsilon \varepsilon - |\eta|. \]
We want to show that the solution $k_{t,\text{ren}}^{(\varepsilon)}$ converges to $r_t$ which satisfied new hierarchy $=$: \textit{Vlasov hierarchy}

\[
\begin{aligned}
\frac{dr_t}{dt} &= V \Delta r_t \\
|r_t|_{t=0} &= r_0
\end{aligned}
\]

This equation describes an evolution of a \textit{virtual interacting particle system} appearing in the Vlasov limit.

In general, this evolution is not related with a new Markov generator.
Consider the case of an initial Poisson measure:

\[ r_0(\eta) = e^\lambda(\rho_0, \eta). \]

Under some general conditions, the scaling leads to solution \( r_t \) of the same form:

\[ r_t(\eta) = e^\lambda(\rho_t, \eta), \quad \eta \in \Gamma_0. \]

The Vlasov hierarchical equation in this case implies a non-linear equation for \( \rho_t \):

\[ \frac{\partial}{\partial t} \rho_t(x) = \nu(\rho_t)(x), \quad x \in \mathbb{R}^d, \]

which we will call Vlasov-type equation corresponding to the considered Markov evolution.
Birth and death evolutions

Birth-and-death generators

\[ L_{\text{bad}} = L^- + L^+ , \]

where

\[
\begin{align*}
(L^- F) (\gamma) & := \sum_{x \in \gamma} d(x, \gamma \setminus x) \left[ F(\gamma \setminus x) - F(\gamma) \right] , \\
(L^+ F) (\gamma) & := \int_{\mathbb{R}^d} b(x, \gamma) \left[ F(\gamma \cup x) - F(\gamma) \right] dx
\end{align*}
\]
Denote $L^{-} = L^{-}(d)$, $L^{+} = L^{+}(b)$.

We will use scaling of rates $b, d$, say, $b_{\varepsilon}, d_{\varepsilon}$, correspondingly, $\varepsilon > 0$.

Scaling of $L_{\text{bad}}$

\[ L_{\text{bad}, \varepsilon} = L^{-}(d_{\varepsilon}) + \varepsilon^{-1}L^{+}(b_{\varepsilon}), \]

General conditions for the weak convergence of $L_{\varepsilon}^{\triangle}$ to the limiting Vlasov generator $V^{\triangle}$ considered in Finkelshtein/K/Kutoviy, 2009, to appear in JSP.

Below we consider some examples from this paper.
Birth-and-death systems

Example (Contact model = branching with mortality)

\[(LF)(\gamma) = m \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)] + \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) [F(\gamma \cup y) - F(\gamma)] dy;\]

\[(VE): \quad \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) + \lambda (\rho_t \ast a)(x).\]

Scaling:

\[\lambda \mapsto \varepsilon^{-1} \lambda, \quad a \mapsto \varepsilon a\]
Example (Migration model)

\[(LF)(\gamma) = \sum_{x \in \gamma} \sum_{y \in \gamma \setminus x} a(x - y)[F(\gamma \setminus x) - F(\gamma)] + \sigma \int_{\mathbb{R}^d} [F(\gamma \cup x) - F(\gamma)]dx;\]

\[(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = -\rho_t(x)(\rho_t \ast a)(x) + \sigma.\]

Scaling:

\[a \mapsto \varepsilon a\]

\[\sigma \mapsto \varepsilon^{-1} \sigma\]
**Example (Bolker-Pacala model in spatial ecology)**

\[
(LF)(\gamma) = \sum_{x \in \gamma} \left( m + \sum_{y \in \gamma \setminus x} a^{-}(x - y) \right) [F(\gamma \setminus x) - F(\gamma)] \\
+ \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^{+}(x - y) [F(\gamma \cup y) - F(\gamma)] dy;
\]

\[
(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) - \rho_t(x)(\rho_t * a^{-})(x) + (\rho_t * a^{+})(x).
\]

Scaling:

\[
a^{-} \mapsto \varepsilon a^{-} \\
a^{+} \mapsto \varepsilon^{-1} \varepsilon a^{+} = a^{+}.
\]

That is a non-local Fisher-Kolmogorov equation, see, e.g., Coville/Dupaigne, 2008.
Example (Ecological model with establishment)

\[
(LF)(\gamma) = m \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)] \\
+ \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) e^{-\sum_{u \in \gamma} \phi(y-u)} [F(\gamma \cup y) - F(\gamma)] dy;
\]

(VE) : \[\frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) + \lambda (a * \rho_t)(x) e^{-(\phi * \rho_t)(x)}\]

Scaling:

\[a \mapsto \varepsilon a, \quad \phi \mapsto \varepsilon \phi\]

\[\lambda \mapsto \varepsilon^{-1} \lambda\]

This equation is new one and shall be analyzed in detail.
Example (Ecological model with fecundity)

\[
(LF)(\gamma) = m \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)] \\
+ \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x-y) e^{-\sum_{u \in \gamma \setminus x} \phi(x-u)} [F(\gamma \cup y) - F(\gamma)] dy;
\]

(VE): \[ \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) + \lambda (a \ast (\rho_t e^{-(\phi \ast \rho_t)}))(x). \]

Scaling:

\[ a \mapsto \varepsilon a, \quad \phi \mapsto \varepsilon \phi \]
\[ \lambda \mapsto \varepsilon^{-1} \lambda \]
Example (Dieckmann--Law model)

\[(LF)(\gamma) = \sum_{x \in \gamma} \left( m + \sum_{y \in \gamma \setminus x} a^-(x - y) \right) [F(\gamma \setminus x) - F(\gamma)] + \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x - y) \left( \lambda + \sum_{u \in \gamma \setminus x} b(x - u) \right) [F(\gamma \cup y) - F(\gamma)] dy; \]

\[(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) - \rho_t(x)(\rho_t \ast a^-)(x) + \lambda(\rho_t \ast a^+)(x) + (((b \ast \rho_t) \rho_t) \ast a^+)(x). \]

Scaling:

\[a^- \mapsto \varepsilon a^-, \quad a^+ \mapsto \varepsilon^{-1} \varepsilon a^+ = a^+, \quad b \mapsto \varepsilon b.\]
Conservative particle systems

Example (Free Kawasaki)

\[(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) [F(\gamma \setminus x \cup y) - F(\gamma)] dy;\]

\[(VE): \quad \frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a)(x) - \rho_t(x) \langle a \rangle.\]
Example (Density dependent Kawasaki)

\[(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) \sum_{u \in \gamma} b(x, y, u) [F(\gamma \setminus x \cup y) - F(\gamma)] \, dy;\]

\[
\frac{\partial}{\partial t} \rho_t(x) = \int_{\mathbb{R}^d} \rho_t(y) a(x - y) \int_{\mathbb{R}^d} \rho_t(u) b(y, x, u) \, du \, dy
- \rho_t(x) \int_{\mathbb{R}^d} a(x - y) \int_{\mathbb{R}^d} \rho_t(u) b(x, y, u) \, du \, dy.
\]

In particular, if \(b(x, y, u) = b(x - u)\) then

\[(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = ((\rho_t(\rho_t \ast b)) \ast a)(x) - \langle a \rangle \rho_t(x) (\rho_t \ast b)(x).\]

If \(b(x, y, u) = b(y - u)\) then

\[
\frac{\partial}{\partial t} \rho_t(x) = (\rho_t \ast b)(x) (\rho_t \ast a)(x) - \rho_t(x) (\rho_t \ast a \ast b)(x).
\]
Example (Gibbs–Kawasaki)

\[(LF') (\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) e^{-E^\phi(x, \gamma)} [F(\gamma \setminus x \cup y) - F(\gamma)] \, dy;\]

\[(\text{VE}): \quad \frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a)(x) \exp \left\{ - (\rho_t * \phi)(x) \right\}
- \rho_t(x) (a * \exp \left\{ -\rho_t * \phi \right\})(x).\]

Scaling:

\[\phi \mapsto \varepsilon \phi\]
Example (Gradient diffusion)

\[
(LF)(\gamma) = \sum_{x \in \gamma} \Delta_x F(\gamma) - \sum_{x \in \gamma} \sum_{y \in \gamma \setminus x} \langle \nabla \phi(x - y), \nabla_x F \rangle
\]

\[
(VE) \quad \frac{\partial}{\partial t} \rho_t(x) = \Delta \rho_t(x) - \int \phi(x - y) \langle \nabla \rho_t(x), \nabla \rho_t(y) \rangle \, dy
\]

\[
- \rho_t(x) \int \langle \nabla \phi(x - y), \nabla \rho_t(y) \rangle \, dy
\]

Scaling:

\[
\phi \mapsto \varepsilon \phi
\]

Convergence problem

In all models above the weak convergence of the generators

\[ L_\varepsilon^\Delta \to V^\Delta, \quad \varepsilon \to 0 \]

is shown.

A difficult question: convergence of solutions of hierarchical equations to the Vlasov hierarchy solution.

In each model even just the problem of the existence for the solution to hierarchy need a special analysis and difficult technical work (non-equilibrium dynamics).

But we need essentially more:
detailed control of the solution uniformly w.r.t. parameters on any finite time interval.

Two regimes:
zero density systems = \( L^1 \) solutions to be considered
positive density systems = \( L^\infty \) solutions
For the positive density case only known results (Finkelshtein/K/Kutoviy 2009-2010) concern

- Contact Model
- Migration Model
- Glauber Dynamics
- Bolker-Pacala model in spatial ecology

Zero density case should be possible to analyze also in other models and under more relaxing conditions on parameters (work in progress).
Two component density dependent ecological model

$$(LF)(\gamma^+, \gamma^-) := \sum_{x \in \gamma^+} \left( m^+ + \sum_{x' \in \gamma^+ \setminus x} a^-_1 (x - x') + \sum_{y \in \gamma^-} b^-_1 (x - y) \right)$$

$$\times \left[ F(\gamma^+ \setminus x, \gamma^-) - F(\gamma^+, \gamma^-) \right]$$

$$+ \sum_{y \in \gamma^-} \left( m^- + \sum_{y' \in \gamma^- \setminus y} a^-_2 (y - y') + \sum_{x \in \gamma^+} b^-_2 (x - y) \right)$$

$$\times \left[ F(\gamma^+, \gamma^- \setminus y) - F(\gamma^+, \gamma^-) \right]$$
+ \sum_{x' \in \gamma^+} \int_{\mathbb{R}^d} \left( a_1^+ (x - x') + \sum_{y \in \gamma^-} b_1^+ (y, x, x') \right) \\
\left[ F(\gamma^+ \cup x, \gamma^-) - F(\gamma^+, \gamma^-) \right] dx \\
+ \sum_{y' \in \gamma^-} \int_{\mathbb{R}^d} \left( a_2^+ (y - y') + \sum_{x \in \gamma^+} b_2^+ (x, y, y') \right) \\
\left[ F(\gamma^+, \gamma^- \cup y) - F(\gamma^+, \gamma^-) \right] dy.
Scaling:

\[ a_{1,\varepsilon}^{\pm} = \varepsilon a_{1}^{\pm}, \quad b_{1,2,\varepsilon}^{-} = \varepsilon b_{1,2}^{-} \]

\[ b_{1,2,\varepsilon}^{+} = \varepsilon^{2} b_{1,2}^{+} \]

Birth intensity:

\[ 1 \mapsto \varepsilon^{-1} \]
Vlasov equation in ecology

\[
\frac{\partial}{\partial t}\rho^+_t (x) = -m^+ \rho^+_t (x) - \rho^+_t (x) (a^-_1 \ast \rho^+_t) (x) \\
- \rho^+_t (x) (\rho^-_t \ast b^-_1) (x) + (\rho^+_t \ast a^+_1) (x) \\
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho^+_t (x') \rho^-_t (y) b^+_1 (y, x, x') \, dx' \, dy
\]

and

\[
\frac{\partial}{\partial t}\rho^-_t (y) = -m^- \rho^-_t (y) - \rho^-_t (y) (a^+_1 \ast \rho^-_t) (y) \\
- \rho^-_t (y) (\rho^+_t \ast b^+_1) (y) + (\rho^-_t \ast a^-_1) (y) \\
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho^-_t (x) \rho^+_t (y') b^+_1 (x, y, y') \, dx \, dy'
\]
Hydrodynamic scaling

Initial state $\mu_0^{(\varepsilon)}$ s.t.

$$k_{0, \text{ren}}^{(\varepsilon)}(\eta) := \varepsilon^{d|\eta|} k_0^{(\varepsilon)}(\eta) \to r_0(\eta), \quad \varepsilon \to 0, \ \eta \in \Gamma_0,$$

Scaled potentials:

$$U \mapsto U(\varepsilon^{-1} \cdot)$$

In the case of BAD-processes we need scaled (accelerated) intensity of birth:

$$z \mapsto \varepsilon^{-d} z$$

Corresponding generator for corr. functions

$$L_{\varepsilon}^{\text{hd}}$$
Rescaled hierarchy

\[
\begin{align*}
\frac{d k_{t, \text{ren}}^{(\varepsilon)}}{dt} &= L_{\varepsilon, \text{ren}}^{\text{hd}} k_{t, \text{ren}}^{(\varepsilon)} \\
\left. k_{t, \text{ren}}^{(\varepsilon)} \right|_{t=0} &= k_{0, \text{ren}}^{(\varepsilon)}
\end{align*}
\]

where

\[
L_{\varepsilon, \text{ren}}^{\text{hd}} = \varepsilon^2 d|\eta| L_{\varepsilon}^{\text{hd}} - d|\eta|.
\]

We want to show that the solution \( k_{t, \text{ren}}^{(\varepsilon)} \) converges to \( r_t \) which satisfied

**HD hierarchy**

\[
\begin{align*}
\frac{dr_t}{dt} &= L^{\text{hd}} r_t \\
\left. r_t \right|_{t=0} &= r_0
\end{align*}
\]

This equation describes an evolution of a virtual interacting particle system appearing in the HD limit.
Consider the case of an initial Poisson measure:

\[ r_0(\eta) = e^{\lambda(\rho_0, \eta)}. \]

Under general conditions, the scaling leads to the solution \( r_t \) of the same form:

\[ r_t(\eta) = e^{\lambda(\rho_t, \eta)}, \quad \eta \in \Gamma_0. \]

(chaos preservation property)

The HD hierarchical equation in this case implies a non-linear equation for \( \rho_t \): 

\[ \frac{\partial}{\partial t} \rho_t(x) = \nu(\rho_t)(x), \quad x \in \mathbb{R}^d, \]

which we will call \textit{hydrodynamic equation} (HDE) corresponding to the considered Markov evolution.
Example (Plankton model)

\[(LF)(\gamma) = \sum_{x \in \gamma} \Delta x F(\gamma) + \sum_{x \in \gamma} \left( m + \sum_{y \in \gamma \setminus x} a^- (x - y) \right) [F(\gamma \setminus x) - F(\gamma)] + \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x - y) [F(\gamma \cup y) - F(\gamma)] dy; \]

\[(HDE) : \frac{\partial}{\partial t} \rho_t(x) = \Delta \rho_t(x) - m \rho_t(x) - \kappa^- \rho_t^2(x) + \kappa^+ \rho_t(x) \]

Fisher-KPP equation.
Scaling:

\[a^- \mapsto a^- (\varepsilon^{-1}) \]
\[a^+ \mapsto \varepsilon^{-d} a^+ (\varepsilon^{-1}) \]
Mixed scaling in Bolker-Pacala model

\[(LF)(\gamma) = \sum_{x \in \gamma} \left( m + \sum_{y \in \gamma \setminus x} a^-(x - y) \right) [F(\gamma \setminus x) - F(\gamma)] \]

\[+ \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x - y) [F(\gamma \cup y) - F(\gamma)] dy; \]

Vlasov scaling for dispersion (birth) part
and HD scaling for competition (death) part of generator
Scaling:

\[ a^- \mapsto a^- (\varepsilon^{-1}) \]
\[ a^+ \mapsto \varepsilon^{-d} \varepsilon^d a^+ = a^+ \].

(HDE) : \[ \frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a^+)(x) - m\rho_t(x) - \kappa^- \rho_t^2(x) = \]
\[ \int a^+(x-y)(\rho_t(y) - \rho_t(x))dy + \kappa^+ \rho_t(x) - m\rho_t(x) - \kappa^- \rho_t^2(x). \]

That is a non-local Fisher-Kolmogorov equation.