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Geometric approach to filtering: some illustrations

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Intertwined diffusions

$$p : \mathbf{R}^k \to R^n$$

smooth, onto;

$\mathcal{A}$,$\mathcal{B}$ diffusion operators on $\mathbf{R}^n$, $\mathbf{R}^k$ s.t.

$$\mathcal{B}(f \circ p) = \mathcal{A}(f) \circ p \qquad f : \mathbf{R}^n \to \mathbf{R}$$

$$\mathcal{A}(f)(x) = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \sum b^i(x) \frac{\partial}{\partial x^i}.$$

Symbol of $\mathcal{A}$

$$\sigma_x^{\mathcal{A}} : (\mathbf{R}^n)^* \to \mathbf{R}^n$$

$$\sigma_x^{\mathcal{A}}(a_1, \ldots, a_n)^i := \sum a^{ij}(x) a_j$$

diffusion = semi-elliptic : $\sigma_x^{\mathcal{A}} \geq 0 \iff \{a^{i.j}(x)\}_{i,j} \geq 0$

Intertwined symbols

$$\begin{tikzcd}
(\mathbf{R}^k)^* \arrow[r, "\sigma^{\mathcal{B}}"] & \mathbf{R}^k \arrow[d, "Dp(u)"] \\
(R^n)^* \arrow[u, "Dp(u)^*"] \arrow[r, "\sigma^{\mathcal{A}}"] & \mathbf{R}^n
\end{tikzcd}$$

commutes

Horizontal Lift

$$
\begin{tikzcd}
(\mathbf{R}^k)^* \arrow[r, "\sigma^{\mathcal{B}}"] & \mathbf{R}^k \arrow[d] \\
(\mathbf{R}^n)^* \arrow[u, "Dp(u)^*"] \arrow[r, "\sigma^{\mathcal{A}}"] & E_{p(u)} \arrow[u, bend right, "h_u"']
\end{tikzcd}
$$

unique.

Horizontal lifts of vector fields, paths, and diffusion operators

For $A : \mathbf{R}^n \to \mathbf{R}^n$ such that $A(x) \in E_x$ define $A^H : \mathbf{R}^k \to R^k$ by

$$A^H(u) := h_u(A(p(u)).$$

For cohesive $\mathcal{A} = \sum X^j X^j + A$ define

$$\mathcal{A}^H := \sum X^{j,H} X^{j,H} + A^H.$$

Cohesive diffusion operators

$\mathcal{A}$ is cohesive if

- $\sigma^{\mathcal{A}}$ has constant rank,
  (giving $E \subset \mathbf{R}^n \times \mathbf{R}^n \quad E = \{(x, v) : v \in E_x\}$).
- For a representation

$$\mathcal{A} = \sum X^j X^j + A$$

all X^j and A are 'E-valued'.

Decomposition of diffusion operators

$$\mathcal{B} = \mathcal{A}^H + \mathcal{B}^V$$

decomposition into horizontal and vertical diffusion operators

unique

SDE's

$\mathcal{A}$,$\mathcal{B}$ given by SDE's:

$$dx_t = X(x_t)\, dB_t + A(x_t)\, dt, \qquad du_t = \tilde{X}(u_t)\, d\tilde{B}_t + \tilde{A}(u_t)\, dt$$

$$X : \mathbf{R}^n \to \mathbf{L}(R^m; \mathbf{R}^n) \qquad A : \mathbf{R}^n \to \mathbf{R}^n$$

$$\tilde{X} : \mathbf{R}^k \to \mathbf{L}(\mathbf{R}^{\tilde{m}}; \mathbf{R}^k) \qquad \tilde{A} : \mathbf{R}^k \to \mathbf{R}^k$$

$\sigma_x^{\mathcal{A}} = X(x) \circ X(x)^*$ so $X(x)[\mathbf{R}^m] = E_x$.

$$Y_x : E_x \to \mathbf{R}^m$$

Set $Y_x := (X(x))^{-1} : E_x \to \mathbf{R}^m$
pseudo-inverse.

Using induced $\langle -, - \rangle_x$ by $X(x)$ on E_x

$$Y_x = (X(x))^*.$$

$$X(x)Y_x = Id \qquad Y_x X(x) = K^{\perp}(x) : \mathbf{R}^m \to \ker X(x)^{\perp}.$$

The lift via SDE's

$\exists! \ell_u$ with $\ker \ell_u = \ker X(p(u))$ such that

$$\begin{tikzcd}
(\mathbf{R}^k)^* \arrow[r, "\tilde{X}(u)^*"] & \mathbf{R}^{\tilde{m}} \arrow[r, "\tilde{X}(u)"] & \mathbf{R}^k \arrow[d, "Dp(u)"'] \\
(\mathbf{R}^n)* \arrow[u, "(Dp(u))^*"] \arrow[r, "X(p(u))^*"] & \mathbf{R}^m \arrow[u, "\ell_u"] \arrow[r, "X(p(u))"] & \mathbf{R}^n
\end{tikzcd}$$

commutes. Consequently $h_u = \tilde{X}(u)\ell_u Y_{p(u)}$.

Horizontal lift of SDE

$$dx_t^H = X^H(x_t^H)dB_t + A^H(x_t^H)dt + \text{STRAT TERM}$$

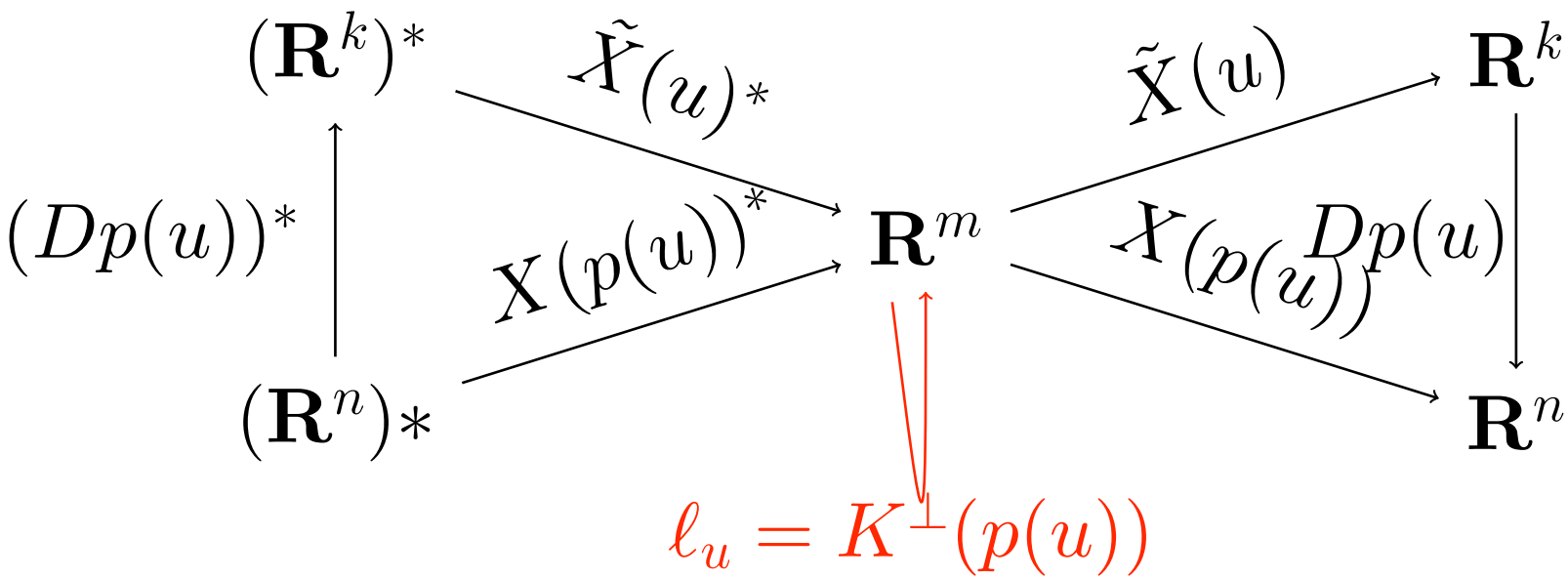
$$X^H(u) := h_u\left(X(p(u))\right) : \mathbf{R}^m \to \mathbf{R}^k$$

Since $h_u = \tilde{X}(u)\ell_u Y_{p(u)}$.

$$X^H(u)(e) = \tilde{X}(u)\ell_u\left(K^\perp(p(u))(e)\right)$$

If SDE's are p-related

$$m = \tilde{m}, \quad B = \tilde{B}, \quad X(p(u))e = Dp(u)\left(\tilde{X}(u)e\right)$$



Redundant noise: El&Yor '93

$\exists /\!\!/_t(x_.) \in O(m)$ with $/\!\!/_t(x_.)[ker X(x_0)] = \ker X(x_t)$

$$B^e_t := \int_0^t /\!\!/_s(x_.)^{-1} K^\perp(x_s)\, dB_s$$

$$\beta_t := \int_0^t /\!\!/_s(x_.)^{-1} K(x_s)\, dB_s$$

B^e, β indep. BM's on $\ker^\perp X(x_0)$, $\ker X(x_t)$.

$\mathcal{F}^{B^e} = \mathcal{F}^{x_.}$

Decomposition of noise

$$B_t^e := \int_0^t /\!\!/_s(x_.)^{-1} K^{\perp}(x_s)\, dB_s$$

$$\beta_t := \int_0^t /\!\!/_s(x_.)^{-1} K(x_s)\, dB_s$$

$$dB_t = /\!\!/_t(x_.)\, dB_t^e + /\!\!/_t(x_.) d\beta_t$$

Decomposition of Ito SDE's

$$du_t = \tilde{X}(u_t)\, dB_t + \tilde{A}(u_t)\, dt$$

$$= \tilde{X}(u_t)/\!/_t\, dB_t^e + \tilde{A}(u_t)\, dt + \tilde{X}(u_t)/\!/_t\, d\beta_t$$

$$= X^H(u_t)\, dB_t + \tilde{A}(u_t)\, dt + \tilde{X}(u_t)/\!/_t\, d\beta_t$$

$$= X^H(u_t)\, dB_t + A^H(u_t) dt$$
$$+ [\tilde{A}(u_t) - A^H(u_t)]\, dt + \tilde{X}(u_t)/\!/_t\, d\beta_t$$

Decomposition of Stratonovich SDE's

$$du_t = \tilde{X}(u_t) \circ dB_t + \tilde{A}(u_t)\, dt$$

$$= X^H(u_t) \circ dB_t + A^H(u_t)\, dt$$
$$+ \tilde{X}(u_t) K(p(u_t)) \circ dB_t + [\tilde{A}(u_t) - A^H(u_t)]\, dt$$

This gives:

$$du_t =$$

$$h_{u_t} \circ dx_t + [\tilde{A}(u_t) - A^H(u_t)]\, dt + \tilde{X}(u_t) \circ\ d \int_0^t /\!/_s\, d\beta_s$$

Therefore for $f : \mathbf{R}^k \to \mathbf{R}$:

$$f(u_t) = f(u_0) + \int_0^t Df(u_s)\ h_{u_s} \circ dx_s$$

$$+ \int_0^t \mathcal{B}^V(f)(u_s)\ ds + \int_0^t Df(u_s)\tilde{X}(u_s)/\!/_s\, d\beta_s$$

Conditioning

For $f : \mathbf{R}^k \to \mathbf{R}$ defining $\pi_t(f) : \mathbf{R}^k \to \mathbf{R}$ by:

$$\pi_t(f)(u_0) = \mathbf{E}\left\{ f(u_t) | x_s : 0 \leq s \leq t \right\}$$ gives

$$\pi_t(f)(u_0) = f(u_0) + \int_0^t \pi_s \left(Df(-)(h_-) \right)(u_0) \circ dx_s$$

$$+ \int_0^t \pi_s \left(B^V(f) \right)(u_0)\, ds$$

Another Decomposition

Assume SDE's are p-related. They induce Hilbert spaces of vector fields, $\tilde{H}$ on $\mathbf{R}^k$ and H on $\mathbf{R}^n$, with $\quad p_* : \tilde{H} \to H, \quad \tilde{X}(-)e \mapsto X(-)e.$

Set $(H^H)^\perp = \ker p_*$, all vertical vector fields. We get:

$$\tilde{H} = H^H \bigoplus (H^H)^\perp$$

$$\mathcal{B} = \mathbb{A} + \mathbb{B}^V.$$

Classical Filtering

"Signal process" $(z_t, 0 \leqslant t \leqslant T)$ on $\mathbf{R}^d$ s.t.

$$dz_t = V(z_t, x_t)dW_t + \beta(z_t, x_t)dt$$

"Observation process", $(x_t, 0 \leqslant t \leqslant T)$ in $\mathbf{R}^n$ s.t.:

$$dx_t = X^{(1)}(x_t)dB_t + X^{(2)}(x_t)dW_t + b(z_t, x_t)dt.$$

Take $\mathbf{R}^k = \mathbf{R}^d \times \mathbf{R}^n$, p the projection.
Set $u_t = (z_t, x_t)$.

Change of Probability

Remove the horizontal lift of the offending drift term by the Girsanov-Maruyama theorem.

$$\tilde{\mathbf{P}} = Z_T \mathbf{P}$$

Kallianpur-Striebel formula

A version of Bayes' formula:

$$\pi_t(f)(u_0) = \frac{\tilde{\pi}_t(f Z_t^{-1})(u_0)}{\tilde{\pi}_t(Z_t^{-1})(u_0)} \qquad \mathbf{P}_{u_0} - as.$$

SPDE Problem

For a solution u_t of an SPDE, and a point x_0, find the conditional law of u_t given

1. $\{u_s(x_0) : 0 \leq s \leq t\}$.

2. $\{\int u_s(x)\, dx : 0 \leq s \leq t\}$.

A simple example of (2)

For $u_t : S^1 \to \mathbf{R}$, with:

$$du_t(\theta) = \Delta u_t(\theta) + u_t(\theta) dW_t + \phi(\theta) dB_t$$

"Observation": $x_t := \int_{S^1} u_t(\theta)\, d\theta$

$$dx_t = x_t dW_t + m dB_t$$

Signal space

For smooth initial u_0 can consider u_t in some smooth Sobolev space H^s.

Take $p(u) := \int_{S^1} u(\theta)\, d\theta$.

Assume $m \neq 0$.

Horizontal lift map

Horizontal lift $h_u : \mathbf{R} \to H^s$ given by

$$h_u(\lambda) == \frac{p(u)u + m\phi}{p(u)^2 + m^2}\lambda.$$

Filtering equation

Recall:

$$\pi_t(f)(u_0) = f(u_0) + \int_0^t \pi_s\left(Df(-)(h_-)\right)(u_0) \circ dx_s$$

$$+ \int_0^t \pi_s\left(B^V(f)\right)(u_0)\, ds$$

Take $f = \delta_z$ so

$$\pi_t(f) = \mathbf{E}\{u_t(z) | \int_{S^1} u_s(\theta)\, d\theta, 0 \leq s \leq t\}$$

Filtering equation ctd.

We obtain:

$$\pi_t(\delta_z)(u_0) = u_0(z) + \int_0^t \pi_s(\delta_z \circ \Delta)(u_0)ds$$
$$+ \int_0^t \frac{p\pi_s(\delta_z)+m\phi(z)}{p^2+m^2}(x_s dW_s + m dB_s).$$

Decomposition method

$$du_t = h_{u_t}(1)dx_t + du_t - h_{u_t}(1)dx_t$$
$$= \frac{x_t u_t + m\phi}{x_t^2 + m^2} dx_t + \Delta u_t dt + dM_t$$

where

$$\begin{aligned} dM_t &= u_t dW_t - \frac{x_t^2 u_t + m x_t \phi}{x_t^2 + m^2} dW_t + \phi dB_t - \frac{x_t u_t m + m^2 \phi}{x_t^2 + m^2} \\ &= \frac{m^2 u_t - m x_t \phi}{x_t^2 + m^2} dW_t + \frac{-x_t u_t m + x_t^2 \phi}{x_t^2 + m^2} dB_t. \end{aligned}$$

Decomposition method conclusion

For $\bar{u}_t = \mathbf{E}\{u_t | \sigma(x_s, 0 \leq s \leq t)\}$

$$d\bar{u}_t = \frac{x_t \bar{u}_t + m\phi}{x_t^2 + m^2} dx_t + \Delta \bar{u}_t dt.$$

Note: $\bar{u}_t(z) = \pi_t(\delta_z)(u)$

Stochastic Flows

$x_0 \in M$ M compact smooth manifold

$p : \mathrm{Diff}(M) \to M \qquad p(\xi) = \xi(x_0)$

$\xi_t \in \mathrm{Diff}(M)$ a stochastic flow

the solution flow of an SDE:

$dx_t = X(x_t) \circ \, dB_t + A(x_t)dt,$

assumed cohesive.

Skew-product decomposition

$\mathrm{Diff}_{x_0}(M) := \{\xi \in \mathrm{Diff}(M) : \xi(x_0) = x_0\}$ acts on right of $\mathrm{Diff}(M)$ by composition. Get:

$$\xi_t = \tilde{x}_t g_t^{x_\cdot} \qquad , 0 \leq t < \infty,$$

with $\tilde{x}_\cdot$ the horizontal lift from id and $g_\cdot^\sigma$ $\mathrm{Diff}_{x_0}(M)$-valued, independent of $x_\cdot$, for $\mathbf{P}_{x_0}^{\mathcal{A}}$-almost all $\sigma : [0, \infty) \to M$.

Formula for the horizontal lift of the one-point motion

$\tilde{x}_{\cdot}$ is the solution flow to the SDE:

$$dz_t = X(z_t)Y(x_t) \circ dx_t.$$

Formula for the vertical process

Take $A = 0$ then:

$$dg_t^\sigma(x) = T\tilde{\sigma}_t^{-1}\Big(X(\tilde{\sigma}_t g_t^\sigma(x))(K(\sigma_t) \circ dB_t)\Big).$$

Application: gradient estimates

$$P_t(f)(x) := \mathbf{E} f(\xi_t(x))$$

Therefore:

$$\begin{aligned} d(P_t(f))x_0(v) &= \mathbf{E} df_{x_t} T_{x_0} \xi_t(v) \\ &= \mathbf{E}\Big\{ df_{x_t} \mathbf{E}\big\{ T_{x_0}\xi_t(v) | x_s : 0 \leq s \leq t \big\} \Big\}. \end{aligned}$$

THAT’S IT THANKS!

A simple example

For $u_t : [0,1] \to \mathbf{R}$, Dirichlet b.c. with:

$$du_t(x) = \triangle u_t(x)dt + \sum_{i=1}^{m} \Phi_i(x, u_t(x))dB_t^i.$$

"Observation": $y_t := u_t(x_0)$

$$dy_t = (\Delta u_t)(x_0)dt + \sum_{i=1}^{m} \Phi_i(x_0, y_t)dB_t^i.$$

Signal space

For smooth initial u_0 can consider u_t in some smooth Sobolev space. Take $p(u) := u(x_0)$.

Markovianisation

Remove $h_u(\Delta u(x_0))$ to get new probability measure $\tilde{\mathbf{P}}$.

$$h_u(\triangle u(x_0)) = \frac{\sum_{j=1}^{m} \Phi_j(x_0, u(x_0))\Phi_j(x_0, u(x))}{\sum_{k=1}^{n} (\Phi_k(x_0, u(x_0)))^2} \triangle u(x_0).$$

Under $\tilde{\mathbb{P}}$

$$du_t(x) = [\frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2}\Phi_i(x_0, u_t(x_0))\ d\tilde{B}_t^i]$$

$$+[\left(\triangle u_t(x) - \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2}\triangle u_t(x_0)\right) dt$$

$$+\sum_{i=1}^m \left(\Phi_i(x, u_t(x)) - \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2}\Phi_i(x_0, u_t(x_0))\right) d\tilde{B}_t^i].$$

$$
\begin{aligned}
du_t(x) &= [\frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2}\, d(u_t(x_0))] \\
&+[\left(\triangle u_t(x) - \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2} \triangle u_t(x_0) \right) dt \\
&+ \sum_{i=1}^m \left(\Phi_i(x, u_t(x)) - \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\Phi_j(x, u_t(x))}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2} \Phi_i(x_0, u_t(x_0)) \right) d\tilde{B}_t^i].
\end{aligned}
$$

Conditioning

Set $\overline{u_t}(x) := \tilde{\mathbf{E}}\{u_t(x)|u_s(x_0) : 0 \leq s < \infty\}$ etc.

Then:

$$d\overline{u_t}(x) = \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\overline{\Phi_j(x, u_t(x))}}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2}\Phi_i(x_0, u_t(x_0))\, d\tilde{B}_t^i$$
$$+ \left(\overline{\triangle u_t}(x) - \frac{\sum_{j=1}^m \Phi_j(x_0, u_t(x_0))\overline{\Phi_j(x, u_t(x))}}{\sum_{k=1}^n (\Phi_k(x_0, u_t(x_0)))^2}\triangle u_t(x_0)\right) dt$$

Formula for $\overline{Du_t(x_0)}$ rather nicer.