Ergodic measures for Markov semigroups

Tomasz Szarek

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Basic definitions

Let \((X, \rho)\) be a Polish space. Let \(\mathcal{B}(X)\) be the space of all Borel subsets of \(X\) and let \(\mathcal{B}_b(X)\) (resp. \(\mathcal{C}_b(X)\)) be the Banach space of all bounded, measurable (resp. continuous) functions on \(X\) equipped with the supremum norm \(\| \cdot \|_\infty\). We denote by \(\text{Lip}_b(X)\) the space of all bounded Lipschitz continuous functions on \(X\).

Let \((P_t)_{t \geq 0}\) be the transition semigroup of a Markov family \(Z = ((Z^x(t))_{t \geq 0}, x \in X)\) taking values in \(X\).

We shall assume that the semigroup \((P_t)_{t \geq 0}\) is \textit{Feller}, i.e. \(P_t(\mathcal{C}_b(X)) \subset \mathcal{C}_b(X)\) and that the Markov family is \textit{stochastically continuous}, which implies that:

\[
\lim_{t \to 0^+} P_t \psi(x) = \psi(x) \text{ for all } x \in X \text{ and } \psi \in \mathcal{C}_b(X).
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We say that a transition semigroup \((P_t)_{t \geq 0}\) has the \textit{e-property} at \(x \in X\) if the family of functions \((P_t \psi)_{t \geq 0}\) is equicontinuous at \(x\) for any bounded and Lipschitz continuous function \(\psi\).

The semigroup \((P_t)_{t \geq 0}\) has the e-property if the above condition holds at any \(x \in X\).

Let \((P^*_t)_{t \geq 0}\) be the dual semigroup defined on the space of all Borel probability measures \(\mathcal{M}_1\) by the formula

\[
P^*_t \mu(B) := \int_X P_t 1_B d\mu \quad \text{for} \ B \in \mathcal{B}(X).
\]

Recall that \(\mu^* \in \mathcal{M}_1\) is \textit{invariant} for the semigroup \((P_t)_{t \geq 0}\) (or the Markov family \((Z^x(t))_{t \geq 0}\)) if \(P^*_t \mu^* = \mu^*\) for all \(t \geq 0\).
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The set $\mathcal{I}$

For a given $T > 0$ and $\mu \in \mathcal{M}_1$ define

$$Q^T \mu := T^{-1} \int_0^T P_s^* \mu ds.$$ 

We write $Q^T(x, \cdot)$ in the particular case when $\mu = \delta_x$. The crucial role is played by the set

$$\mathcal{I} := \{ x \in X : \text{the family of measures } (Q^t(x))_{t \geq 0} \text{ is tight} \}.$$ 

Obviously, if $\mathcal{I} \neq \emptyset$, then the semigroup $(P_t)_{t \geq 0}$ admits an invariant measure. Moreover, if $\mu_*$ is an invariant measure, then $\text{supp } \mu_* \subset \mathcal{I}$. 
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We start with recalling the following classical theorem.

**Theorem 1.**

Assume that \((P_t)_{t \geq 0}\) is a Feller semigroup. If there exist a compact set \(K \subset X\) and a point \(x \in X\) such that

\[
\limsup_{T \to \infty} Q^T(x, K) > 0,
\]

then \((P_t)_{t \geq 0}\) admits an invariant measure.
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Theorem 2. (A. Lasota and T.S.)

Let \((P_t)_{t \geq 0}\) be a Feller semigroup. Assume that there exists a point \(z \in X\) such that for every \(\delta > 0\)

\[
\limsup_{T \to \infty} Q^T(x, B(z, \delta)) > 0 \quad \text{for some } x \in X.
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If the semigroup \((P_t)_{t \geq 0}\) has the e-property in \(z \in X\), then \(z \in T\). Consequently, \((P_t)_{t \geq 0}\) admits an invariant measure.
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Theorem 3. (T. Komorowski, S. Peszat and T.S.)

Assume that \((P_t)_{t \geq 0}\) has the e–property and that there exists a point \(z \in X\) such that for every \(\delta > 0\) and every \(x \in X\)

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\limsup_{T \to \infty} Q^T(x, B(z, \delta)) > 0.
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Then \((P_t)_{t \geq 0}\) admits a unique invariant measure \(\mu_*\). Moreover,

\[
\operatorname{w-lim}_{t \to \infty} Q^t \nu = \mu_*,
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for any \(\nu \in \mathcal{M}_1\) that is supported in \(T\).
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We say that \( x \in X \) generates a measure \( \mu \in \mathcal{M}_1 \) if

\[
\mu \in \text{cl conv}\{Q^t(x) : t \geq 0\},
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where the closure is taken in the weak topology.

An invariant measure \( \mu \in \mathcal{M}_1 \) is called \textbf{ergodic} if every \( A \in \mathcal{B}(X) \) such that \( P_t 1_A = 1_A \) for \( t \geq 0 \) satisfies \( \mu(A) \in \{0, 1\} \).
Uniqueness of invariant measures

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Proposition 1. (R. Kapica, M. Śleczka, D. Worm and T.S.)

Assume that \((P_t)_{t \geq 0}\) has the e–property.

- If \(x \in \mathcal{T}\), then the sequence \((Q^t(x))_{t \geq 0}\) weakly converges to some invariant measure.
- If \(\mu \in \mathcal{M}_1\) is generated by \(x \in \mathcal{T}\), then the sequence \((Q^t\mu)_{t \geq 0}\) has the same limit as \((Q^t(x))_{t \geq 0}\).

As a consequence of this proposition we obtain the following theorem.

Theorem 4. (R. Kapica, M. Śleczka and T.S.)

Let \((P_t)_{t \geq 0}\) be a Feller semigroup admitting two distinct ergodic measures, say \(\mu\) and \(\nu\). If \((P_t)_{t \geq 0}\) has the e–property, then \(\text{supp} \mu \cap \text{supp} \nu = \emptyset\).
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We say that a Markov semigroup \((P_t)_{t \geq 0}\) is **weakly topologically irreducible** if for all \(x_1, x_2 \in X\) there is \(y \in X\) such that for any open set \(A \ni y\) there exist \(t_1, t_2 > 0\) with \(P_{t_i}1_A(x_i) > 0\) for \(i = 1, 2\).

**Proposition 1.**

Let \((P_t)_{t \geq 0}\) be weakly topologically irreducible. If \((P_t)_{t \geq 0}\) has the e-property, then it has at most one invariant measure.
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**Proposition 1.**

Let \((P_t)_{t \geq 0}\) be weakly topologically irreducible. If \((P_t)_{t \geq 0}\) has the e-property, then it has at most one invariant measure.
Existence of many invariant measures

We shall assume the following concentrating condition:

(C) There exists a compact set \( K \subset X \) such that for any \( \varepsilon > 0 \) and every \( x \in X \)

\[
\limsup_{T \to +\infty} Q^T(x, K^\varepsilon) > 0,
\]

where \( K^\varepsilon = \{ x \in X : \inf_{y \in K} \rho(x, y < \varepsilon) \} \).

If \( (P_t)_{t \geq 0} \) satisfies the e-property and \( x \in T \), then by \( \nu^x \) we denote the weak limit of \( (Q^t(x))_{t \geq 0} \).
Existence of many invariant measures

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If $(P_t)_{t \geq 0}$ satisfies the e-property and $x \in T$, then by $\nu^x$ we denote the weak limit of $(Q^t(x))_{t \geq 0}$. 
We may formulate the following result.

**Theorem 4. (D. Worm and T.S.)**

If \((P_t)_{t \geq 0}\) satisfies the e-property and \((C)\), then there exists a Borel set \(K_0 \subset K \cap \mathcal{T}\) such that

- \(x \in \text{supp} \nu^x\) for all \(x \in K_0\),
- if \(x, y \in K_0\) with \(x \neq y\), then \(\nu^x \neq \nu^y\),
- for every ergodic measure \(\mu\) there is an \(x \in K_0\) such that \(\mu = \nu^x\).
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Existence of many invariant measures

Now we fix $x_0 \in X$. For $f : X \rightarrow \mathbb{R}$ and $\theta > 0$ we define the local Lipschitz constant

$$|f|_{Lip, \theta} := \sup \left\{ \frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y; x, y \in B(x_0, \theta) \right\}.$$ 

Proposition 2. (D. Worm and T.S.)

Let $(P_t)_{t \geq 0}$ satisfy the e-property and (C). If there are sequences $t_n > 0$ and $\delta_n \downarrow 0$ and a non-decreasing function $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that for all bounded and Lipschitz functions and $\theta > 0$

$$|P_{t_n} f|_{Lip, \theta} \leq C(\theta)(\|f\|_\infty + \delta_n \text{ Lip } f).$$ 

Then $(P_t)_{t \geq 0}$ admits only finitely many ergodic measures.
Existence of many invariant measures

Now we fix $x_0 \in X$. For $f : X \to \mathbb{R}$ and $\theta > 0$ we define the local Lipschitz constant

$$|f|_{Lip,\theta} := \sup \left\{ \frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y; x, y \in B(x_0, \theta) \right\}.$$

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$$|P_{t_n}f|_{Lip,\theta} \leq C(\theta)(\|f\|_\infty + \delta_n \text{Lip } f).$$

Then $(P_t)_{t \geq 0}$ admits only finitely many ergodic measures.
Existence of many invariant measures

Now we fix \( x_0 \in X \). For \( f : X \to \mathbb{R} \) and \( \theta > 0 \) we define the local Lipschitz constant

\[
|f|_{\text{Lip},\theta} := \sup \left\{ \frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y; x, y \in B(x_0, \theta) \right\}.
\]

Proposition 2. (D. Worm and T.S.)

Let \( (P_t)_{t \geq 0} \) satisfy the e-property and \((C)\). If there are sequences \( t_n > 0 \) and \( \delta_n \downarrow 0 \) and a non-decreasing function \( C : \mathbb{R}_+ \to \mathbb{R}_+ \), such that for all bounded and Lipschitz functions and \( \theta > 0 \)

\[
|P_{t_n}f|_{\text{Lip},\theta} \leq C(\theta)(\|f\|_{\infty} + \delta_n \text{Lip } f).
\]

Then \( (P_t)_{t \geq 0} \) admits only finitely many ergodic measures.
Applications to SPDE’s

We study the Markov process defined by the stochastic evolution equation

\[ dZ(t) = (AZ(t) + F(Z(t))) \, dt + RdW(t). \]  

(1)

Here \( A \) is the generator of a \( C_0 \)-semigroup \( S = (S(t))_{t \geq 0} \) acting on some real separable Hilbert space \( X \), \( F \) maps (not necessarily continuously) \( D(F) \subset X \) into \( X \), \( R \) is a bounded linear operator from another Hilbert space \( \mathcal{H} \) to \( X \), and \( W = (W(t))_{t \geq 0} \) is a cylindrical Wiener process on \( \mathcal{H} \) defined over a certain filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \).
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Applications to SPDE’s

We study the Markov process defined by the stochastic evolution equation

\[ dZ(t) = (AZ(t) + F(Z(t))) \, dt + RdW(t). \]  \hspace{1cm} (1)

Here \( A \) is the generator of a \( C_0 \)-semigroup \( S = (S(t))_{t \geq 0} \) acting on some real separable Hilbert space \( \mathcal{X} \), \( F \) maps (not necessarily continuously) \( D(F) \subset \mathcal{X} \) into \( \mathcal{X} \), \( R \) is a bounded linear operator from another Hilbert space \( \mathcal{H} \) to \( \mathcal{X} \), and \( W = (W(t))_{t \geq 0} \) is a cylindrical Wiener process on \( \mathcal{H} \) defined over a certain filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \).
We suppose that for every $x \in \mathcal{X}$ there is a unique mild solution $Z^x = (Z^x_t)_{t \geq 0}$ of (1) starting from $x$, and that (1) defines in that way a Markov family. We assume that for any $x \in \mathcal{X}$, the process $Z^x$ is stochastically continuous. The corresponding transition semigroup is given by

$$P_t \psi(x) = \mathbb{E} \psi(Z^x(t)),$$

$\psi \in B_b(\mathcal{X})$, and we assume that it is Feller. A function $\Phi : \mathcal{X} \mapsto [0, +\infty)$ will be called a Lyapunov function, if it is measurable and

$$\lim_{\|x\|_\mathcal{X} \to \infty} \Phi(x) = \infty.$$
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We shall assume that the deterministic equation

$$\frac{dY(t)}{dt} = AY(t) + F(Y(t)), \quad Y(0) = x \quad (2)$$

defines a continuous semi-dynamical system

$$Y^x = (Y^x(t), \ t \geq 0).$$

A set $\mathcal{K} \subset \mathcal{X}$ is called a global attractor for $Y^x$ if

1) it is invariant under the semi-dynamical system, i.e.

$$Y^x(t) \in \mathcal{K} \text{ for any } x \in \mathcal{K} \text{ and } t \geq 0.$$

2) for any $\varepsilon, R > 0$ there exists $T$ such that

$$Y^x(t) \in \mathcal{K} + \varepsilon B(0, 1) \text{ for } t \geq T \text{ and } \|x\|_{\mathcal{X}} \leq R.$$
Applications to SPDE’s

We shall assume that the deterministic equation

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A set \( K \subset X \) is called a **global attractor** for \( Y^x \) if

1) it is invariant under the semi-dynamical system, i.e.
\[ Y^x(t) \in K \text{ for any } x \in K \text{ and } t \geq 0. \]

2) for any \( \varepsilon, R > 0 \) there exists \( T \) such that
\[ Y^x(t) \in K + \varepsilon B(0, 1) \text{ for } t \geq T \text{ and } \|x\|_X \leq R. \]
The family \((Z^x(t))_{t \geq 0}, x \in \mathcal{X}\), is \textbf{stochastically stable} if for every \(\varepsilon, R, t > 0\)

\[
\inf_{x \in B(0,R)} P \left( \|Z^x(t) - Y^x(t)\|_{\mathcal{X}} < \varepsilon \right) > 0.
\]
Theorem 5. (T. Komorowski, S. Peszat and T.S.)

Assume that:

- there exists a global attractor $\mathcal{K}$ of the semi-dynamical system $(Y^x(t), \ t \geq 0)$ defined by (2);
- there exists a certain Lyapunov function $\Phi$ such that
  \[
  \sup_{t \geq 0} \mathbb{E} \Phi(Z^x(t)) < \infty \quad \text{for any } x \in \mathcal{X},
  \]
- the family $(Z^x(t))_{t \geq 0}$, $x \in \mathcal{X}$, is stochastically stable, its transition semigroup has the e-property and
  \[
  \bigcap_{x \in \mathcal{K}} \bigcup_{t \geq 0} \Gamma^t(x) \neq \emptyset, \quad (3)
  \]

where $\Gamma^t(x) = \text{supp } P^*_t \delta_x$.
Then, \( (Z^x(t))_{t \geq 0}, \ x \in \mathcal{X} \) admits a unique invariant measure \( \mu_* \). Moreover, we have

\[
\lim_{t \to \infty} Q^t \mu = \mu_*
\]

for any \( \mu \in M_1 \).

If we additionally assume that the attractor \( \mathcal{K} \) is a singleton, then \( (P_t)_{t \geq 0} \) is asymptotically stable, i.e.

\[
\lim_{t \to \infty} P_t^* \mu = \mu_*
\]

for any \( \mu \in M_1 \).
Theorem 5. (continuation)

Then, \((Z^x(t))_{t \geq 0}, x \in \mathcal{X}\) admits a unique invariant measure \(\mu_*\). Moreover, we have

\[
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If we additionally assume that the attractor \(\mathcal{K}\) is a singleton, then \((P_t)_{t \geq 0}\) is asymptotically stable, i.e.

\[
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\]

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