Confidence Intervals for Quantiles When Applying Variance-Reduction Techniques

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   - Bahadur Representations

2 Variance-Reduction Techniques for Quantiles
   - Framework for VRTs
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   - Confidence Intervals for Quantile with VRT

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Quantiles

- Random variable $X$ with CDF $F$.
- For $0 < p < 1$, the $p$-quantile of $F$ is

$$\xi_p = F^{-1}(p) \equiv \inf\{x : F(x) \geq p\}$$

**Applications**
- Project planning
- Value-at-risk in finance
Confidence Intervals (CIs) for Quantiles

- Typical approach to estimate $\xi_p = F^{-1}(p)$ using simulation
  1. Generate $n$ samples, and compute $\hat{F}_n$ as estimator of $F$.
  - e.g., Crude Monte Carlo (CMC): $X_1, \ldots, X_n \sim F$ i.i.d., and
    \[
    \hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)
    \]
  2. Estimator of $\xi_p$ is $\hat{\xi}_{p,n} = \hat{F}_n^{-1}(p)$
Typical approach to estimate $\xi_p = F^{-1}(p)$ using simulation

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2. Estimator of $\xi_p$ is $\hat{\xi}_{p,n} = \hat{F}_n^{-1}(p)$

CI for $\xi_p$ gives measure of error of estimator $\hat{\xi}_{p,n}$

1. Prove estimator $\hat{\xi}_{p,n}$ satisfies CLT
   \[
   \sqrt{n}(\hat{\xi}_{p,n} - \xi_p) \Rightarrow N(0, \kappa_p^2)
   \]
   - CMC: $\kappa_p = \sqrt{p(1-p)/f(\xi_p)}$, where $f = F'$.
2. Provide consistent estimator $\hat{\kappa}_{p,n}$ for $\kappa_p$ to get 95% CI for $\xi_p$:
   \[
   \left(\hat{\xi}_{p,n} \pm 1.96 \frac{\hat{\kappa}_{p,n}}{\sqrt{n}}\right)
   \]
Confidence Intervals (CIs) for Quantiles

- When using crude Monte Carlo (CMC), CI may be wide
  - Especially when $p \approx 0$ or $p \approx 1$.
- Variance-reduction techniques (VRTs) for quantiles
  - Importance sampling (IS): Glynn (1996)
  - IS and stratified sampling: Glasserman et al. (2000)
- Above papers prove CLTs for VRT quantile estimators.
  - None gives consistent estimator of variance $\kappa_p^2$ in CLT.
- We provide consistent estimators of $\kappa_p^2$ to get CI when using VRT.
  - Approach based on Bahadur-Ghosh representation.
Bahadur Representation for CMC

Heuristic argument:

- Suppose $f(\xi_p) > 0$, where $f = F'$.
- CMC: $\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)$
- $\hat{F}_n \approx F$ so $\hat{\xi}_p, n \approx \xi_p$.
- Since $p = F(\xi_p)$,

\[
p \approx F(\hat{\xi}_p, n)
\]
\[
\approx F(\xi_p) + f(\xi_p)(\hat{\xi}_p, n - \xi_p)
\]
\[
\approx \hat{F}_n(\xi_p) + f(\xi_p)(\hat{\xi}_p, n - \xi_p).
\]

Thus,

\[
\hat{\xi}_p, n \approx \xi_p - \frac{\hat{F}_n(\xi_p) - p}{f(\xi_p)}.
\]
Bahadur Representation for CMC

- **Recall:**
  \[ \hat{\xi}_{p,n} \approx \xi_p - \frac{\hat{F}_n(\xi_p) - p}{f(\xi_p)}. \]

- Can make this rigorous for CMC
  - Let \( p_n = p + O(n^{-1/2}) \) and \( \hat{\xi}_{p_n,n} = \hat{F}_n^{-1}(p_n) \)
  - Write
    \[ \hat{\xi}_{p_n,n} = \xi_p - \frac{\hat{F}_n(\xi_p) - p_n}{f(\xi_p)} + R_n \]

- Bahadur (1966): If \( f(\xi_p) > 0 \) and \( f'(\xi_p) \) exists,
  \[ R_n = O(n^{-3/4}(\log n)^{1/2}(\log \log n)^{1/4}) \text{ a.s.} \]

- Ghosh (1971): If \( f(\xi_p) > 0 \),
  \[ \sqrt{n}R_n \Rightarrow 0. \]
Assumptions for VRTs

- Let $\hat{F}_n$ be estimated CDF using VRT.

**Assumptions**

A1. $P\{\hat{F}_n(x) \text{ is monotonically increasing in } x\} \to 1$ as $n \to \infty$.

A2. For every $a_n = O(n^{-1/2})$,

$$\sqrt{n} \left[ (F(\xi_p + a_n) - F(\xi_p)) - (\hat{F}_n(\xi_p + a_n) - \hat{F}_n(\xi_p)) \right] \Rightarrow 0, \text{ as } n \to \infty.$$

A3. $\sqrt{n} \left[ \hat{F}_n(\xi_p) - F(\xi_p) \right] \Rightarrow N(0, \psi_p^2)$ as $n \to \infty$ for some $0 < \psi_p < \infty$.

- A1–A3 hold (under certain moment conditions) for
  - Combined importance sampling and stratified sampling (IS+SS)
  - Antithetic variates
  - Control variates
Example: Importance Sampling (IS), Glynn (1996)

- $F_*$ is another CDF such that $F$ is absolutely cont. wrt $F_*$.  
- $L(u) = dF(u)/dF_*(u)$ is likelihood ratio.  
- $E_*$ is expectation under $F_*$, so 
  \[
  F(x) = \int I(u \leq x) \, dF(u) \\
  = \int I(u \leq x) \, L(u) \, dF_*(u) = E_*[ I(X \leq x) \, L ].
  \]

- IS: generate $X_1, L_1, \ldots, X_n, L_n$ i.i.d. using $F_*$. 
- IS estimator of CDF is 
  \[
  \hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x) \, L_i
  \]

- IS estimator of $p$-quantile is $\hat{\xi}_{p,n} = \hat{F}_n^{-1}(p)$. 
- A1–A3 hold if $E_*[ I(X \leq \xi_p + \delta) \, L^{2+\epsilon} ] < \infty$ for some $\epsilon, \delta > 0$.  

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Bahadur-Ghosh Representation for VRTs

Theorem

- Suppose \( f(\xi_p) > 0 \) and VRT CDF estimator \( \hat{F}_n \) satisfies A1–A3.
- Then \( \hat{\xi}_{p_n,n} = \hat{F}_n^{-1}(p_n) \) with \( p_n = p + O(n^{-1/2}) \) satisfies

\[
\hat{\xi}_{p_n,n} = \xi_p - \frac{\hat{F}_n(\xi_p) - p_n}{f(\xi_p)} + R_n,
\]

where

\[
\sqrt{nR_n} \Rightarrow 0.
\]

- Sun and Hong (2010) prove a.s. Bahadur representation for IS
  - Stronger assumptions
  - For fixed \( p_n \equiv p \).
Corollary

VRT quantile estimator $\hat{\xi}_{p,n} = \hat{F}_n^{-1}(p)$ satisfies CLT:

$$\sqrt{n}(\hat{\xi}_{p,n} - \xi_p) \Rightarrow N(0, \kappa_p^2)$$

where

$$\kappa_p = \frac{\psi_p}{f(\xi_p)}$$

and $\psi_p^2$ is variance constant in CLT for $\hat{F}_n(\xi_p)$.

- To construct CI, want consistent estimators for $\psi_p$ and $f(\xi_p)$
  - Straightforward to construct consistent estimate of $\psi_p$. 
Estimating $f(\xi_p)$

- Recall: CLT variance $\kappa_p^2$, where $\kappa_p = \frac{\psi_p}{f(\xi_p)}$
- Glynn (1996): “Major challenge” is estimating $f(\xi_p)$.
- Glasserman et al. (2000): Estimating $f(\xi_p)$ is “difficult”.
- We develop consistent estimate of $\phi_p \equiv 1/f(\xi_p)$.
  - Chain rule of calculus:
    $$\frac{d}{dp} F^{-1}(p) = \frac{1}{f(\xi_p)} = \phi_p.$$  
  - Finite difference: for any $c \neq 0$ (“smoothing parameter”),
    $$\hat{\phi}_{p,n}(c) = \frac{\hat{F}_n^{-1}(p + cn^{-1/2}) - \hat{F}_n^{-1}(p - cn^{-1/2})}{2cn^{-1/2}}$$
  - Bloch-Gastwirth (1968) prove $\tilde{\phi}_{p,n}(c)$ is consistent for CMC
    - Proof uses $X_i = F^{-1}(U_i)$, where $U_i \sim \text{unif}[0, 1]$
    - Proof does not generalize to VRT.
\( \hat{\phi}_{p,n}(c) \) is consistent estimator for \( \phi_p = 1/f(\xi_p) \)

**Corollary**

\( \hat{\phi}_{p,n}(c) \Rightarrow \phi_p \) as \( n \to \infty \) for any \( c \neq 0 \).

**Proof.**

1. By Bahadur-Ghosh representation,

\[
\hat{F}_n^{-1}(p \pm cn^{-1/2}) = \xi_p - \frac{\hat{F}_n(p) - (p \pm cn^{-1/2})}{f(\xi_p)} + R_n,\pm
\]

2. Since \( \sqrt{n}R_{n,\pm} \Rightarrow 0 \),

\[
\hat{\phi}_{p,n}(c) = \frac{\hat{F}_n^{-1}(p + cn^{-1/2}) - \hat{F}_n^{-1}(p - cn^{-1/2})}{2cn^{-1/2}}
\]

\[
= \frac{1}{f(\xi_p)} + \frac{\sqrt{n}}{2c}(R_{n,+} - R_{n,-}) \Rightarrow \frac{1}{f(\xi_p)}
\]
Stochastic Activity Network (SAN)

- Activity durations are i.i.d. exponential with mean 1.
- Estimate 0.95-quantile of longest path from $s$ to $t$.
- IS+SS
  - IS: Mixture of expo-tilted distns [Juneja et al. (2006)]
  - IS: Choose tilting parameters as in Glynn (1996).
  - SS: Path with largest mean is stratification variable.
Empirical Results

Stochastic Activity Network (SAN)

Nominal 90% CIs
Estimated coverage from $10^4$ replications
$c$ is smoothing parameter in finite difference.
Concluding Remarks

- Developed general framework to construct asymptotically valid confidence intervals for quantile with VRT
  - IS+SS
  - Antithetic variates
  - Control variates
- Approach based on Bahadur-Ghosh representation for quantile estimator with VRT.