Asymptotic results for a class of stochastic RDEs with fast transport term and noise acting on the boundary

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joint work with Mark Freidlin

Stochastic Partial Differential Equations (SPDEs):
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The problem

Consider the equation

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,x) = & \dfrac{1}{\epsilon}\mathcal{A}u_\epsilon(t,x) + f(t,x,u_\epsilon(t,x)) \\ & +g(t,x,u_\epsilon(t,x))\dfrac{\partial w^Q}{\partial t}(t,x), \qquad t \geq 0, \quad x \in D, \\ \dfrac{1}{\epsilon}\dfrac{\partial u_\epsilon}{\partial \nu}(t,x) = & \sigma(t,x)\dfrac{\partial w^B}{\partial t}(t,x), \qquad t \geq 0, \quad x \in \partial D, \\ u_\epsilon(0,x) = & u^0(x), \qquad x \in D, \end{cases} \tag{1}$$

depending on a small parameter $\epsilon > 0$.

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- $\mathcal{A}$ is a uniformly elliptic second order operator (in divergence form) and $\partial/\partial\nu$ is the corresponding co-normal derivative.
- The coefficients f, g, σ are measurable and satisfy few other conditions.
- The noisy perturbations are given by two cylindrical Wiener processes w^Q and w^B which are defined respectively on $L^2(D)$ and $L^2(\partial D)$ and have respectively covariance operators Q and B.

  In space dimension $d = 1$, we have $B \in \mathcal{L}(\mathbb{R}^2)$ and we can take $Q = \text{Id}$.

This class of equations describes typically the evolution of concentrations of various components in chemical reactions where

- the concentration is not constant in space, due to spatial transport,
- randomness in the rates of reaction has to be taken into account.

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In these systems the rates of chemical reactions and the diffusion coefficients may have, and as a rule have, different orders: some of them are much smaller than others.

In our work we consider a diffusion rate much larger than the reaction rate and a noisy reaction occurring also on the boundary.

We prove that in this case an averaging-type result holds, so that the SPDE's can be replaced by a suitable one-dimensional SDE. We also study fluctuations.

The averaging result

Assume that the diffusion associated with $\mathcal{A}$ has a unique invariant measure μ, with exponential convergence to equilibrium.
Then for any $T, \delta > 0$ and $p \geq 1$

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [\delta, T]} \left(\int_D |u_\epsilon(t,x) - v(t)|^2 \, \mu(dx) \right)^{\frac{p}{2}} = 0,$$

where $v(t)$ is the solution of the problem

$$\begin{cases} dv(t) = \hat{F}(t, v(t))\, dt + \hat{G}(t, v(t)) dw^Q(t) + \hat{\Sigma}(t) dw^B(t) \\ \\ v(0) = \displaystyle\int_D u^0(x)\, \mu(dx), \end{cases}$$

whose coefficients and noisy perturbations are obtained by averaging the coefficients and the noises in equation (1) with respect to μ.

The limit v does not depend on the space variable x and it coincides in law with the solution of the one-dimensional SDE

$$\begin{cases} dv(t) = \Phi(t, v(t))\, dt + \Psi(t, v(t))\, dw_t \\ \\ v(0) = \int_D u^0(x)\, \mu(dx) \end{cases}$$

where w_t is a standard Brownian motion and

$$\Phi, \Psi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$$

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For the drift we have

$$\Phi(t, v) := \int_D f(t, x, v)\, \mu(dx), \qquad (t, v) \in [0, \infty) \times \mathbb{R}$$

The expression of the diffusion coefficient is more complicate.
In the simpler case

$$\mu(dx) = \frac{1}{|D|}\,dx \qquad (\mathcal{A} \text{ self-adjoint}),$$

we have

$$\Psi(t,v) = \frac{1}{|D|}\left(\int_D |\,[Q\,g(t,\cdot,v)]\,(x)|^2\,dx + \int_{\partial D} |\,[B\,\sigma(t,\cdot)]\,(\eta)|^2\,d\eta\right)^{\frac{1}{2}}.$$

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In the case $d=1$, we get

$$\Psi(t,v) = \frac{1}{|D|}\left(\int_D g^2(t,x,v)\,dx + \int_{\partial D} \sigma^2(t,\eta)\,d\eta\right)^{\frac{1}{2}}.$$

Fluctuations of u_ϵ from v

Then, we study the weak limit of

$$z_\epsilon(t) := \frac{u_\epsilon(t) - v(t)}{\sqrt{\epsilon}}, \qquad t \geq 0,$$

as $\epsilon \downarrow 0$.

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Then, we study the weak limit of

$$z_\epsilon(t) := \frac{u_\epsilon(t) - v(t)}{\sqrt{\epsilon}}, \qquad t \geq 0,$$

as $\epsilon \downarrow 0$.

We assume that g does not depend on the state (additive noise) and we show that for any $t > 0$

$$z_\epsilon(t) \rightharpoonup I_0(t), \ \ \text{in } L^2(D, \mu), \qquad \epsilon \downarrow 0,$$

where $I_0(t, x)$ is a Gaussian random field taking values in $L^2(D, \mu)$, which is explicitly given.

In the case there is no noise acting on the boundary

$$I_0(t,x) := \int_0^\infty e^{sA}\Pi[g(t,\cdot)dw^Q(s,\cdot)](x)$$

$$= \sum_{k=1}^\infty \int_0^\infty e^{sA}\Pi[g(t,\cdot)Qe_k](x)\, d\beta_k(s),$$

where

$$\Pi h(x) = h(x) - \int_D h(y)\,\mu(dy).$$

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Notice that if g does not depend on t, the weak limit of $z_\epsilon(t)$ does not depend on t.

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$$\mathcal{A}(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i},$$

with

$$\inf_{x \in D} \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq a_0 \sum_{i=1}^{d} \xi_i^2, \qquad \xi \in \mathbb{R}^d,$$

for some $a_0 > 0$. The coefficients a_{ij} and b_i are smooth.

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for some $a_0 > 0$. The coefficients a_{ij} and b_i are smooth.

In what follows we shall set

$$H := L^2(D), \qquad Z := L^2(\partial D).$$

We shall denote by A the realization in H of the operator $\mathcal{A}$, endowed with the boundary condition

$$\frac{\partial}{\partial \nu}(x) := \langle a(x)\nu(x), \nabla \rangle_{\mathbb{R}^d} = 0, \qquad x \in \partial D. \tag{2}$$

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Hypothesis

The semigroup e^{tA} admits a unique invariant measure μ and there exists some $\gamma > 0$ such that for any $h \in L^2(D, \mu)$ and $t \geq 0$

$$\int_D \left|e^{tA}h(x) - \int_D h(y)\mu(dy)\right|^2 \mu(dx) \leq c\, e^{-\gamma t} \int_D |h(x)|^2 \mu(dx).$$

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Due to the invariance of μ and the kernel representation of e^{tA}, it is possible to check that $H \subset H_\mu$, with continuous embedding.

In fact, it is possible to prove that

$$\mu(dx) = m(x)\, dx, \qquad x \in D,$$

for some smooth function (bounded) $m \geq 0$.

Clearly, if $b \equiv 0$, then A is self-adjoint in H. This implies that for a complete orthonormal system $\{e_k\}_{k\geq 0}$ in H and an increasing sequence $\{\alpha_k\}_{k\geq 0} \subset [0, +\infty)$

$$Ae_k = -\alpha_k e_k, \qquad k \in \mathbb{N}.$$

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Let e_0 be the constant eigenfunction corresponding to $\alpha_0 = 0$ and let α_1 be the first positive eigenvalue. Then

$$\mu(dx) = e_0^2\, dx = |D|^{-1}\, dx,$$

and $H = H_\mu$. Moreover, $\gamma = \alpha_1$.

The coefficients

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1. *The mappings* $f, g : [0, \infty) \times D \times \mathbb{R} \to \mathbb{R}$ *are measurable and the mappings*

$$f(t, x, \cdot), g(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$$

are Lipschitz continuous, uniformly w.r.t. $t \in [0, T]$ *and* $x \in D$.

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2. *The mapping* $\sigma : [0,\infty) \times \partial D \to \mathbb{R}$ *is measurable and for any* $T > 0$

$$\sup_{t \in [0,T]} |\sigma(t,\cdot)|_{L^\infty(\partial D)} < \infty.$$

In what follows, for any $t \geq 0$, $h_1, h_2 \in H$ and $x \in D$ we shall set

$$F(t, h_1)(x) := f(t, x, h_1(x)),$$

and

$$[G(t, h_1)h_2](x) := g(t, x, h_1(x))h_2(x),$$

and for any $t \geq 0$, $z \in Z$ and $\eta \in \partial D$ we shall set

$$[\Sigma(t)z](\eta) := \sigma(t, \eta)z(\eta).$$

The noisy terms

We assume that $w^Q(t)$ and $w^B(t)$ are both cylindrical Wiener processes, on H and Z, with covariance operators $Q \in \mathcal{L}^+(H)$ and $B \in \mathcal{L}^+(Z)$, respectively.

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Namely

$$w^Q(t) = \sum_{k\in\mathbb{N}} \lambda_k e_k \beta_k(t), \qquad w^B(t) = \sum_{k\in\mathbb{N}} \theta_k f_k \hat{\beta}_k(t).$$

Here $\{e_k\}_{k\in\mathbb{N}}$ and $\{f_k\}_{k\in\mathbb{N}}$ are orthonormal bases of H and Z with

$$Qe_k = \lambda_k e_k, \qquad Bf_k = \theta_k f_k.$$

$\{\beta_k\}_{k\in\mathbb{N}}$ and $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$ are sequences of independent standard Brownian motions, defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Hypothesis

1. *If* $d \geq 2$*, there exists* $\rho < 2d/(d-2)$ *such that*

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As both $d/(d-2) > 1$ and $d/(d-1) > 1$, neither Q nor B are required to be Hilbert-Schmidt operators.

Moreover, if $d = 1$ we have no conditions on $\{\lambda_k\}$. This means that we can deal with space-time white noise.

The boundary stochastic convolution

For any $\delta \geq \delta_0$ and $h \in Z$, let $N_\delta h$ denote the solution of

$$\left\{\begin{array}{ll}(\delta - \mathcal{A})N_\delta h(x) = 0, & x \in D,\\[10pt] \langle a(x)\nu(x), \nabla N_\delta h(x)\rangle_{\mathbb{R}^d} = h(x), & x \in \partial D.\end{array}\right.$$

N_δ is the *Neumann map* associated with $\mathcal{A}$.

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N_δ is the *Neumann map* associated with $\mathcal{A}$.

It is known that

$$N_\delta \in \mathcal{L}(Z^\alpha, H^{\alpha+\frac{3}{2}}), \quad \alpha \geq 0,$$

(here $H^\alpha := H^\alpha(D)$ and $Z^\alpha := H^\alpha(\partial D)$).

Assume that v is a Z-valued function such that $v(\cdot)$ is twice continuously differentiable and $N_\delta v(0) \in D(A)$. Then, the solution of the problem

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) = \mathcal{A} y(t,x), \quad t \geq 0, \quad x \in D, \\[2ex] \langle a(x)\nu(x), \nabla y(t,x)\rangle_{\mathbb{R}^d} = v(t,x), \quad t \geq 0, \quad x \in \partial D, \\[2ex] y(0,x) = 0, \quad x \in D, \end{cases}$$

is given by

$$y(t) = (\delta - A) \int_0^t e^{(t-s)A} N_\delta v(s)\, ds.$$

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Notice that the formula above can be extended to less regular functions $v(t,x)$.

For any $\epsilon > 0$, consider the problem

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) = \dfrac{1}{\epsilon}\mathcal{A}y(t,x), \quad t \geq 0, \quad x \in D \\ \\ \dfrac{1}{\epsilon}\left\langle a(x)\nu(x), \nabla y(t,x)\right\rangle_{\mathbb{R}^d} = \sigma(t,x)\dfrac{\partial w^B}{\partial t}(t,x), \quad x \in \partial D, \\ \\ y(0,x) = 0, \quad x \in D. \end{cases}$$

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In analogy with the formula above, by taking

$$\frac{A}{\epsilon}, \qquad \delta = \frac{\delta_0}{\epsilon}, \qquad v(t) = \epsilon\,\Sigma(t)\frac{\partial w^B}{\partial t}(t),$$

we say that the process

$$w^{\epsilon}_{A,B}(t) := (\delta_0 - A)\int_0^t e^{(t-s)\frac{A}{\epsilon}} N_{\delta_0}\left[\Sigma(s)dw^B(s)\right], \qquad t \geq 0,$$

is a mild solution.

Theorem

Fix any $u^0 \in H$. Then, for any $T > 0$, $p \geq 1$ and $\epsilon > 0$ there exists a unique adapted process u_ϵ in $L^p(\Omega; C([0,T]; H))$ s.t.

$$u_\epsilon(t) = e^{t\frac{A}{\epsilon}} u^0 + \int_0^t e^{(t-s)\frac{A}{\epsilon}} F(s, u_\epsilon(s))\, ds + w^\epsilon_{A,Q}(u_\epsilon)(t) + w^\epsilon_{A,B}(t),$$

where, for any $u \in L^p(\Omega; C([0,T]; H))$, we denote

$$w^\epsilon_{A,Q}(u)(t) := \int_0^t e^{(t-s)\frac{A}{\epsilon}} G(s, u(s))\, dw^Q(s), \qquad t \geq 0.$$

Moreover

$$\sup_{\epsilon \in (0,1]} \mathbb{E}\, |u_\epsilon|^p_{C([0,T];H)} \leq c_{T,p} \left(1 + |u^0|^p_H\right).$$

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2. For each $\epsilon > 0$ the mapping

$$u \in L^p(\Omega; C([0,T]; H)) \mapsto w^\epsilon_{A,Q}(u) \in L^p(\Omega; C([0,T]; H)),$$

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3. Conclusion. As $F : H \to H$ is Lipschitz-continuous we get existence and uniqueness and due to the previous two steps we get the uniform bound with respect to $\epsilon \in (0,1]$.

The averaged drift coefficient

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Due to the asymptotic behavior of e^{tA}, for any $T > 0$ and $p \geq 1$ and for any $\epsilon \in (0,1]$

$$\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \left(e^{(t-s)\frac{A}{\epsilon}} F(s, u_\epsilon(s)) - \hat{F}(s, u_\epsilon(s)) \right) ds \right|_{H_\mu}^p$$

$$\leq c_{T,p} \left(1 + |u^0|_H^p \right) \epsilon^p.$$

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It is immediate to check that the mapping

$$\hat{G}(t,\cdot) : H_\mu \to H,$$

is well defined and Lipschitz-continuous, uniformly with respect to $t \in [0, T]$, for any $T > 0$.

We prove that for any $T > 0$ and $p \geq 1$ and for any $\epsilon \in (0,1]$

$$\mathbb{E} \sup_{t \in [0,T]} \left| w^{\epsilon}_{A,Q}(u_\epsilon)(t) - \int_0^t \hat{G}(s, u_\epsilon(s)) dw^Q(s) \right|^p_{H_\mu}$$

$$\leq c_{T,p}\, \epsilon^{\lambda_p} \left(1 + |u^0|^p_H\right),$$

for some constant $\lambda_p = \lambda_p(d) > 0$.

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In what follows, for any $u \in L^p(\Omega; C([0, T]; H))$ we set

$$\hat{w}_{A,Q}(u)(t) := \int_0^t \hat{G}(s, u(s)) dw^Q(s).$$

The averaged boundary coefficient

For each $t \geq 0$ and $h \in Z$, we define

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As $N_{\delta_0} \in \mathcal{L}(Z,H)$, $\Sigma(t) \in \mathcal{L}(Z)$ and H is continuously embedded in H_μ, we have that for any $t \geq 0$

$$\hat{\Sigma}(t) : Z \to \mathbb{R}$$

is well defined and is a bounded linear operator.

We prove that for any $T > 0$ and $p \geq 1$ and for any $\epsilon \in (0,1]$

$$\mathbb{E} \sup_{t \in [0,T]} \left| w^{\epsilon}_{A,B}(t) - \int_0^t \hat{\Sigma}(s) dw^B(s) \right|^p_{H_\mu} \leq c_{T,p}\, \epsilon^{\lambda_p},$$

for some constant $\lambda_p = \lambda_p(d) > 0$.

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In what follows, we set

$$\hat{w}_{A,B}(t) := \int_0^t \hat{\Sigma}(s) dw^B(s).$$

The averaged equation

We introduce the one-dimensional stochastic equation

$$\begin{cases} dv(t) = \hat{F}(t, v(t))\, dt + \hat{G}(t, v(t)) dw^Q(t) + \hat{\Sigma}(t) dw^B(t), \\ \\ v(0) = \int_0 u^0(x)\, \mu(dx). \end{cases}$$

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The equation above has a unique adapted solution
$v \in L^p(\Omega; C([0, T]; \mathbb{R}))$, that is

$$v(t) = \int_0 u^0(x)\,\mu(dx) + \int_0^t \hat{F}(s, v(s))\,ds$$

$$+ \int_0^t \hat{G}(s, v(s))dw^Q(s) + \int_0^t \hat{\Sigma}(s)dw^B(s).$$

The averaging result

Theorem

For any $u^0 \in H$, $p \geq 1/2$ *and* $T > 0$ *and for any* $\delta > 0$ *we have*

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [\delta, T]} \left(\int_D |u_\epsilon(t,x) - v(t)|^2 \, \mu(dx) \right)^p = 0.$$

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We have

$$u_\epsilon(t) - v(t) = \left(e^{t\frac{A}{\epsilon}} u^0 - \left\langle u^0, \mu \right\rangle \right) + \int_0^t \left(\hat{F}(s, u_\epsilon(s)) - \hat{F}(s, v(s)) \right) ds$$

$$+ \int_0^t \left(\hat{G}(s, u_\epsilon(s)) - \hat{G}(s, v(s)) \right) dw^Q(s) + R_\epsilon(t),$$

and R_ϵ goes to zero in $L^p(\Omega; C([0, T]; H_\mu))$.

Normal deviations

Hypothesis

1. *$f(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ is of class C^1, with Lipschitz derivative, uniformly w.r.t. $x \in D$ and $t \in [0, T]$.*

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We define

$$\Pi h = h - \int_D h(y)\,\mu(dy), \qquad u\in H_\mu.$$

Theorem

For any $t > 0$

$$z_\epsilon(t,x) \rightharpoonup I_0(t,x), \qquad \epsilon \downarrow 0,$$

in H_μ, *where* $I_0(t,x)$ *is the Gaussian random field defined by*

$$\int_0^t e^{sA}\Pi G(t)dw^Q(s,x) + \int_0^\infty (\delta_0 - A)e^{sA}\Pi N_{\delta_0}\left[\Sigma(t)dw^B(s)\right](x).$$

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Take $g(t,x) = \sigma(t,x) \equiv 1$ and A self-adjoint. Then,

$$I_0(t,x) = \sum_{k=1}^\infty \lambda_k \int_0^{+\infty} e^{-\alpha_k s} e_k(x)\, d\beta_k(s)$$

$$+ \sum_{k=0}^\infty \theta_k \int_0^{+\infty} \sum_{h=1}^\infty e^{-s\alpha_h} \left\langle f_k, e_{h_{|\partial D}} \right\rangle_Z e_h(x)\, d\hat{\beta}_k(s).$$

We define

$$I_G(t) := \int_0^\infty e^{sA}\Pi\, G(t)dw^Q(s)$$

and

$$I_\Sigma(t) := \int_0^\infty (\delta_0 - A)e^{sA}\Pi\, N_{\delta_0}\left[\Sigma(t)dw^B(s)\right].$$

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We can prove

$$\mathbb{E} \int_D |I_G(t,x)|^2\, d\mu(x) < \infty, \qquad t \geq 0$$

and

$$\mathbb{E} \int_D |I_\Sigma(t,x)|^2\, d\mu(x) < \infty, \qquad t \geq 0.$$

Proof of the central limit theorem

For any $t > 0$ we have

$$z_\epsilon(t) = \int_0^t D\hat{F}(s, v(s))z_\epsilon(s)\, ds + I_\epsilon(t) + R_\epsilon(t),$$

where

$$I_\epsilon(t) := \frac{1}{\sqrt{\epsilon}}\left(w^\epsilon_{A,Q}(t) - \hat{w}_{A,Q}(t)\right) + \frac{1}{\sqrt{\epsilon}}\left(w^\epsilon_{A,B}(t) - \hat{w}_{A,B}(t)\right).$$

and

$$R_\epsilon(t) := \frac{1}{\sqrt{\epsilon}}\left(e^{\frac{t}{\epsilon}A}u^0 - \langle u^0, \mu\rangle\right)$$

$$+\frac{1}{\sqrt{\epsilon}}\int_0^t \left(e^{(t-s)\frac{A}{\epsilon}}F(s, u_\epsilon(s)) - \hat{F}(s, u_\epsilon(s))\right) ds$$

$$+\int_0^t \int_0^1 \left[D\hat{F}(s, v(s) + \theta(u_\epsilon(s) - v(s))) - D\hat{F}(s, v(s))\right] z_\epsilon(s)\, d\theta\, ds,$$

- We prove that for any $T > 0$, $t \in (0, T]$ and $\epsilon \in (0, 1]$

$$
\begin{aligned}
&\mathbb{E} \int_D |R_\epsilon(t,x)|^2 \mu(dx) \\
&\leq c_T \left(\frac{1}{\sqrt{\epsilon}} e^{-\frac{\gamma t}{\epsilon}} + \epsilon^\alpha \right) \left(1 + \int_D |u^0(x)|^2 \mu(dx) \right),
\end{aligned}
\tag{3}
$$

for some $\alpha > 0$.

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for some $\alpha > 0$.

- We introduce the problem

$$\zeta_\epsilon(t) = \int_0^t D\hat{F}(s, v(s))\zeta_\epsilon(s)\, ds + I_\epsilon(t),$$

and, as a consequence of (3), we show that for $t > 0$

$$\lim_{\epsilon \to 0} \mathbb{E}\, |z_\epsilon(t) - \zeta_\epsilon(t)|_{H_\mu} = 0.$$

Lemma

For any $t > 0$, it holds

$$\lim_{\epsilon \to 0} \zeta_\epsilon(t) = I_0(t), \quad \text{weakly in } H_\mu. \tag{4}$$

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- For any $x \in D$ and $t > 0$ we can write

$$\zeta_\epsilon(z, x) = \int_0^t H(t, s) K_\epsilon(s)\, ds + I_\epsilon(t, x),$$

where

$$H(t, s) := \exp\left(\int_s^t \int_D f_v(r, x, v(r))\, \mu(dx)\, dr \right)$$

and

$$K_\epsilon(s) := \int_D f_v(s, y, v(s)) I_\epsilon(s, y)\, \mu(dy).$$

- We show that there exists $\gamma > 0$ s.t.

$$\mathbb{E}\left|\int_0^t H(t,s)K_\epsilon(s)\,ds\right|^2 \leq c_t\,\epsilon^\gamma,$$

for $\epsilon \in (0,1]$ and $t \geq 0$.

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- We prove that

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In order to prove (5), with a change of time we have

$$\mathcal{L}(I_\epsilon(t)) = \mathcal{L}(\hat{I}_\epsilon(t)),$$

where

$$\hat{I}_\epsilon(t) := \int_0^{\frac{t}{\epsilon}} e^{rA}\Pi\left[G(t-\epsilon r)dw^Q(r)\right]$$

$$+\int_0^{\frac{t}{\epsilon}}(\delta_0 - A)e^{rA}\Pi\left[N_{\delta_0}\left(\Sigma(t-\epsilon r)dw^B(r)\right)\right].$$

Thus, in order to conclude we have to prove

$$\lim_{\epsilon \to 0} \mathbb{E} \int_D \left| \hat{I}_\epsilon(t,x) - I_0(t,x) \right|^2 \mu(dx) = 0.$$

Step 1: the boundary stochastic convolution

We consider the case $d \geq 2$. By a factorization argument, for any $\alpha \in (1/p, 1)$ we have

$$\mathbb{E} \sup_{t \in [0,T]} |w^{\epsilon}_{A,B}(t)|_H^p \leq c_{T,p,\alpha}$$

$$\times \int_0^T \left(\int_0^s (s-r)^{-2\alpha} \sum_{k \in \mathbb{N}} \theta_k^2 \left| (\delta_0 - A) e^{(s-r)\frac{A}{\epsilon}} N_{\delta_0} \left[\Sigma(r) f_k \right] \right|_H^2 dr \right)^{\frac{p}{2}} ds,$$

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If we define

$$I^\epsilon_k(r) := \left| (\delta_0 - A) e^{(s-r)\frac{A}{\epsilon}} N_{\delta_0} [\Sigma(r) f_k] \right|^2_H,$$

and set $\zeta := \beta/(\beta - 2)$, we have

$$\sum_{k \in \mathbb{N}} \theta_k^2 \, I^\epsilon_k(r) \leq \kappa_B^{\frac{2}{\beta}} \left(\sum_{k \in \mathbb{N}} I^\epsilon_k(r) \right)^{\frac{1}{\zeta}} \sup_{k \in \mathbb{N}} |I^\epsilon_k(r)|^{\frac{\zeta - 1}{\zeta}}.$$

For any $\rho > 0$ we have

$$S_\rho := (\delta_0 - A)^{\frac{3-\rho}{4}} N_{\delta_0} \in \mathcal{L}(Z, H).$$

Then

$$\sum_{k \in \mathbb{N}} I_k^\epsilon(r) = \sum_{k \in \mathbb{N}} \left| e^{\frac{(s-r)}{2} \frac{A}{\epsilon}} (\delta_0 - A)^{\frac{1+\rho}{4}} e^{\frac{(s-r)}{2} \frac{A}{\epsilon}} S_\rho \Sigma(r) f_k \right|_H^2$$

$$= \sum_{h \in \mathbb{N}} \left| \Sigma(r) S_\rho^\star \left[(\delta_0 - A)^{\frac{1+\rho}{4}} e^{\frac{(s-r)}{2} \frac{A}{\epsilon}} \right]^\star e^{\frac{(s-r)}{2} \frac{A^\star}{\epsilon}} e_h \right|_Z^2$$

$$\leq c_{T,\delta} (s-r)^{-\frac{1+\rho}{2}} \sum_{h \in \mathbb{N}} \left| e^{\frac{(s-r)}{2} \frac{A^\star}{\epsilon}} e_h \right|_H^2.$$

Due to the heat kernel representation of e^{tA}, we have

$$\sum_{h\in\mathbb{N}} \left| e^{\frac{(s-r)}{2}\frac{A^\star}{\epsilon}} e_h \right|_H^2 = \int_D \left| k_{\frac{t-s}{2\epsilon}}(\cdot, y) \right|_H^2 dy \le c|D| \left[(s-r)^{-\frac{d}{2}} \vee 1 \right],$$

so that

$$\left(\sum_{k\in\mathbb{N}} I_k^\epsilon(r) \right)^{\frac{1}{\zeta}} \le c_{T,\delta} \left[(s-r)^{-\frac{d+1+\rho}{2\zeta}} \vee 1 \right].$$

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$$\sup_{k\in\mathbb{N}} I_k^\epsilon(r)^{\frac{\zeta-1}{\zeta}} \leq c_{T,\rho}(s-r)^{-\frac{(1+\rho)(\zeta-1)}{2\zeta}}.$$

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Next

$$\sup_{k\in\mathbb{N}}I_k^\epsilon(r)^{\frac{\zeta-1}{\zeta}}\leq c_{T,\rho}(s-r)^{-\frac{(1+\rho)(\zeta-1)}{2\zeta}}.$$

Therefore, for any $\epsilon\in\,(0,1]$

$$\mathbb{E}\sup_{t\in[0,T]}|w_{A,B}^\epsilon(t)|_H^p\leq c_{T,p,\alpha,\rho}\left(\int_0^T\left[s^{-(2\alpha+\frac{d+\zeta}{2\zeta}+\frac{\rho}{2})}\vee 1\right]\,ds\right)^{\frac{p}{2}}.$$

Step 2: the stochastic convolution

For any $u \in L^p(\Omega; C([0,T]; H))$, we denote

$$w^{\epsilon}_{A,Q}(u)(t) := \int_0^t e^{(t-s)\frac{A}{\epsilon}} G(s, u(s))\, dw^Q(s), \qquad t \geq 0.$$

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$$w^\epsilon_{A,Q}(u)(t) := \int_0^t e^{(t-s)\frac{A}{\epsilon}} G(s, u(s))\, dw^Q(s), \qquad t \geq 0.$$

By using arguments analogous to those used for $w^\epsilon_{A,B}$, we have that $w^\epsilon_{A,Q}$ is Lipschitz-continuous from $L^p(\Omega; C([0,T];H)$ into itself, for any $T > 0$ and $p \geq 1$, and

$$\sup_{\epsilon \in (0,1]} \mathbb{E}\, |w^\epsilon_{A,Q}(u)|^p_{C([0,T];H)} \leq c_{T,p} \left(1 + \mathbb{E} \int_0^T |u(s)|^p_H\, ds\right).$$