Stochastic integrals for spde's: a comparison

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Objectives

Compare stochastic integrals used to study spde's:

- Walsh's theory of martingale measures, with extensions
- Krylov & Rozovskii, Da Prato & Zabczyk theory of Hilbert-space-valued stochastic integrals

What can be integrated via each theory?

Do they lead to the same solutions to spde's, or to different solutions?

Familiarity with at least one of these theories is assumed.

Motivation: a comment of Michael Röckner.

Note. Do not discuss Banach-space-valued integrals (e.g. Mikulevicius, Rozovskii, Brzeźniak, van Neerven & Weis).

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Class of spde's

$$c^2 \frac{\partial^2 u(t,x)}{\partial t^2} + \nu \frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) = \sigma(u(t,x))\, \dot{W}_{t,x} + b(u(t,x)),$$
$$t \in \mathbb{R}_+, \; x \in \mathbb{R}^k$$
$$u(0,x) = f(x), \quad \frac{\partial u(0,x)}{\partial t} = v_0(x), \qquad x \in \mathbb{R}^k.$$

$c = 0$, $\nu = 1$: heat equation; $c = 1$, $\nu = 0$: wave equation

Gaussian random forcing $\dot{W}_{t,x}$:

- space time white noise:

$$E(\dot{W}_{t,x}\, \dot{W}_{s,y}) = \delta_0(t-s)\delta_0(x-y)$$

- spatially homogeneous noise:

$$E(\dot{W}_{t,x}\, \dot{W}_{s,y}) = \delta_0(t-s) f(x-y)$$

where $f : \mathbb{R}^k \to \mathbb{R}$ is a function, or a distribution such as δ_0

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Spatially homogeneous noise as a generalized random field

$(W(\varphi),\ \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^k)$, a family of Gaussian random variables,

with covariance functional

$$E(W(\varphi)W(\psi)) = \int_0^\infty dt \int_{\mathbb{R}^k} \Lambda(dx)\,(\varphi(t) * \tilde{\psi}(t))(x),$$
$$= \int_0^\infty dt \int_{\mathbb{R}^k} \mu(d\xi)\,\mathcal{F}\varphi(t)(\xi)\,\overline{\mathcal{F}\psi(t)(\xi)}$$

(μ is the spectral measure: $\Lambda = \mathcal{F}\mu$, $\mu \geqslant 0$, Λ is typically a nonnegative measure on $\mathbb{R}^k$, called the covariance measure)

Typical choices of Λ:

$\Lambda(dx) = f(x)\,dx \qquad \Rightarrow E(W(\varphi)W(\psi)) = \int_0^\infty dt \int_{\mathbb{R}^k} dx \int_{\mathbb{R}^k} dy\, \varphi(t,x) f(x-y)\psi(t,y)$

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Need for stochastic integrals

Let

$$Lu(t,x) = c^2 \frac{\partial^2 u(t,x)}{\partial t^2} + \nu \frac{\partial u(t,x)}{\partial t} - \Delta u(t,x).$$

In order to solve the spde $Lu(t,x) = \sigma(u(t,x))\,\dot{W}_{t,x}$, multiply by a test function and integrate:

$$\int\int \varphi(t,x) Lu(t,x)\,dtdx = \int\int \varphi(t,x)\sigma(u(t,x))\,\dot{W}_{t,x}\,dtdx$$

Meaning of this integral on the right?

Mild formulation.

$$u(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}^k} G(t-r, x-y)\sigma(u(r,y))\,\dot{W}_{r,y}\,drdy,$$

where:

$I_0(t,x)$ is the solution of the homogeneous pde with the same initial conditions;

$G(t-r, x-y)$ is the fundamental solution of this pde.

The (stochastic) integral still needs to be defined.

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Nature of the fundamental solution

Heat equation, all dimensions $k \geqslant 1$:

$$G(t,x) = (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

Wave equation, dimension $k \in \{1,2,3\}$:

$$G_1(t,x) = \frac{1}{2} 1_{\{|x|<t\}}, \quad G_2(t,x) = \frac{1}{2\pi}(t^2 - |x|^2)_+^{-1/2}, \quad G_3(t)(dx) = \frac{1}{4\pi t}\sigma_t(dx),$$

where σ_t is the area measure on $\partial B(0,t)$ (with total mass $4\pi t^2$).

Need: define integrals of the form

$$v_{G,Z}(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-r, x-y)\, Z(r,y)\, \dot{W}_{r,y} dy dr,$$

where $Z(r,y)$ plays the role of $\sigma(u(r,y))$.

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Common stochastic integrals

School A: Walsh, 1986. Integrals with respect to martingale measures.

School B: Krylov & Rozovskii 1979, Da Prato & Zabczyk 1992. Integrals with respect to Hilbert-space-valued Brownian motion.

Related approach: Métivier & Pellaumail 1980. Integral with respect to cylindrical Brownian motion.

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Cylindrical Brownian motion

Let V be a separable Hilbert space and let Q be a symmetric (self-adjoint) and non-negative definite bounded linear operator on V.

Definition

A family of random variables $B = \{B_t(h),\ t \geqslant 0,\ h \in V\}$ *is a cylindrical B.M. on* V *with covariance* Q *if:*

1. for any $h \in V$, $\{B_t(h),\ t \geqslant 0\}$ *defines a B.M. with variance* $t\langle Qh, h\rangle_V$*;*

2. for all $s, t \in \mathbb{R}_+$ *and* $h, g \in V$,

$$E\left(B_s(h)B_t(g)\right) = (s \wedge t)\langle Qh, g\rangle_V,$$

where $s \wedge t := \min(s, t)$*.*

If $Q = \mathrm{Id}_V$*, then* B *is called a standard cylindrical B.M..*

Stochastic integral w.r.t. cylindrical Brownian motion

Let V_Q be the (completion of the) Hilbert space V endowed with the inner-product

$$\langle h, g \rangle_{V_Q} := \langle Qh, g \rangle_V, \qquad h, g \in V.$$

Filtration: $\mathcal{F}_t = \sigma\{B_s(h),\ h \in V,\ 0 \leq s \leq t\}$ (completed).

Let $(v_j)_j$ be a c.o.n.b. of V_Q. For any predictable $g \in L^2(\Omega \times [0, T]; V_Q)$, define:

$$g \cdot B := \sum_{j=1}^{\infty} \int_0^T \langle g_s, v_j \rangle_{V_Q} \, dB_s(v_j). \tag{1}$$

(the series converges in $L^2(\Omega, \mathcal{F}, P)$).

Isometry property:

$$E\left((g \cdot B)^2\right) = E\left(\left(\int_0^T g_s \, dB_s\right)^2\right) = E\left(\int_0^T \|g_s\|_{V_Q}^2 \, ds\right).$$

Note. The integral produces a real-valued random variable.

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Spatially homogeneous noise as a cyl. B.M.

$$E(W(\varphi)W(\psi)) = \int_0^\infty dt \int_{\mathbb{R}^d} \mu(d\xi)\, \mathcal{F}\varphi(t)(\xi)\, \overline{\mathcal{F}\psi(t)(\xi)}.$$

Let U the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ endowed with the semi-inner product

$$\langle \varphi, \psi \rangle_U = \int_{\mathbb{R}^d} \mu(d\xi)\, \mathcal{F}\varphi(\xi)\, \overline{\mathcal{F}\psi(\xi)},$$

$\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, and associated semi-norm $\| \cdot \|_U$. Then U is a separable Hilbert space that may contain Schwartz distributions.

For $\varphi \in U$, define $W_t(\varphi) = W(1_{[0,t]}(\cdot)\varphi(*))$ by approximating $1_{[0,t]}(\cdot)\varphi(*)$ by a sequence $(\psi_n(\cdot, *))$ of elements of $\mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)$.

Proposition

The process $W = \{W_t(\varphi),\, t \geqslant 0,\, \varphi \in U\}$ is a standard cylindrical B.M. ($Q = Id_U$).

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School A: the Walsh integral, 1984

(Worthy) Martingale measure:

$$(M_t(A),\ t \in \mathbb{R}_+,\ A \in \mathcal{B}_b(\mathbb{R}^k)) :$$

$t \mapsto M_t(A)$ is a martingale;
$A \mapsto M_t(A)$ is an $L^2(\Omega, \mathcal{F}, P)$-valued measure.

Covariance measure: a signed measure Q such that

$$E(M_t(A)M_t(B)) = Q([0,t] \times A \times B)$$

Dominating measure: a nonnegative measure K such that

$$|Q([0,t] \times A \times B)| \leqslant K([0,t] \times A \times B)$$

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The Walsh integral

The Walsh stochastic integral

$$\int_{[0,t]\times\mathbb{R}^k} f(t,x)\, M(dt,dx)$$

is defined for predictable random fields $(f(t,x),\ (t,x)\in\mathbb{R}^k)$ such that

$$E\left(\int_{[0,t]\times\mathbb{R}^k\times\mathbb{R}^k} |f(t,x)|\,|f(t,y)|\,K(dt,dx,dy)\right) < \infty$$

Note. The integral produces a real-valued random variable.

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Spatially homogeneous noise as a martingale measure

For rectangles A, let

$$M_t(A) := \lim_{n\to\infty} W(\varphi_n), \qquad \varphi_n \downarrow 1_{[0,t]\times A}, \; \varphi_n \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^k).$$

then extend to bounded Borel sets A by σ-additivity.

Possible integrands: functions $G \in \mathcal{P}_+$, that is, such that

$$\int_0^T ds \int_{\mathbb{R}^k} \Lambda(dx)\,(|G(s,\cdot)| * |G(s,-\cdot)|)(x) < \infty.$$

Applicability for our spde's:

Sufficient for the heat equation in all dimensions and for the wave equation when $k \in \{1,2\}$.

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First extension (D. 1999 and Nualart & Quer-Sardanyons 2007)

This extension still produces integrals that are real-valued random variables.

Possible integrands: the integral

$$\int_0^T \int_{\mathbb{R}^k} G(t-s, x-y) Z(s,y) M(ds, dy)$$

is well-defined provided

$$G(s) \in \mathcal{P}_0^Z \subset \mathcal{S}'(\mathbb{R}^k) \qquad \text{(Schwartz space)},$$

that is, $G(s)$ has rapid decrease, $G(s) \geqslant 0$ unless $Z \equiv 1$, and

$$\int_0^T ds \sup_{x \in \mathbb{R}^k} E(Z(s,x)^2) \int_{\mathbb{R}^k} \mu(d\xi) \, |\mathcal{F}G(s)(\xi)|^2 < \infty.$$

Applicability for our spde's. Sufficient for:

- Heat equation in all dimensions $k \geqslant 1$.
- Linear wave equation in all dimensions $k \geqslant 1$.
- Non-linear wave equation in dimensions $k \in \{1, 2, 3\}$,

First extension (D. 1999 and Nualart & Quer-Sardanyons 2007)

This extension still produces integrals that are real-valued random variables.

Possible integrands: the integral

$$\int_0^T \int_{\mathbb{R}^k} G(t-s, x-y) Z(s,y) M(ds, dy)$$

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Relationships between integrals

Spatially homogeneous noise can be turned into:

(1) a cylindrical Brownian motion;

(2) a worthy martingale measure.

For (1), we have the integral w.r.t. cyl. B.M.
For (2), we have the Walsh stochastic integral, with extension.
What are the relationships?

Proposition

(a) If $g \in \mathcal{P}_+$, then the Walsh-integral $g \cdot W$ is equal to $\int_0^T g_s \, dW_s$.

(b) If $g \in \mathcal{P}_0$, then the Dalang-Nualart-Quer extension $g \cdot W$ of the Walsh integral is equal to $\int_0^T g_s \, dW_s$.

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Further extension (Quer, Sanz-Solé, Nualart, 2004, 2005, 2007)

Hilbert-space-valued integrals (motivated by Malliavin calculus)

Let H be a Hilbert-space, let $U \otimes H$ be the tensor product of U and H. Recall that $X \in U \otimes H$ is essentially a "matrix" such that

$$\sum_{j,k=1}^{\infty} \left(X^{j,k}\right)^2 < +\infty,$$

Formally,

$$X = \sum_{j,k=1}^{\infty} X^{j,k} (e_j \otimes f_k)$$

where (e_j) is a c.o.n.b. of U, (f_k) is a c.o.n.b. of H, $X^{j,k} \in \mathbb{R}$ and

$$\|X\|_{V_Q \otimes H}^2 = \sum_{j,k=1}^{\infty} (X^{j,k})^2.$$

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Extended Walsh theory: H-valued integrals

Possible integrands: predictable $g \in L^2(\Omega \times [0,T]; U \otimes H)$, that is,

$$g_s = \sum_{j,k=1}^{\infty} g_s^{j,k} \, (e_j \otimes f_k) \qquad \text{and} \qquad E\left(\int_0^T \sum_{j,k=1}^{\infty} (g_s^{j,k})^2 \, ds \right) < +\infty.$$

Definition

$$g \cdot W := \sum_{k=1}^{\infty} (g^k \cdot W) \, f_k$$

where

$$g_s^k := \sum_{j=1}^{\infty} g_s^{j,k} e_j, \qquad s \in [0,T].$$

Recall that $g^k \cdot W$ is a W-D-N-Q integral

Result of the integral: an H-valued random vector.

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Examples of integrable H-valued functions

Let Γ be a function defined on $\mathbb{R}_+$ with values in $\mathcal{S}'(\mathbb{R}^k)$ such that, for all $t > 0$, $\Gamma(t)$ is a non-negative distribution with rapid decrease, and

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t)(\xi)|^2 < \infty. \qquad (2)$$

In addition, Γ is a non-negative measure of the form $\Gamma(t, dx)dt$ such that, for all $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} \Gamma(t, \mathbb{R}^k) < \infty.$$

Let $K = \{K(t,x),\, (t,x) \in [0,T] \times \mathbb{R}^k\}$ be a predictable H-valued process with

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|K(t,x)\|_H^p) < \infty, \qquad \text{for some } p \geqslant 2.$$

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Examples of integrable H-valued functions

Theorem 1

Then $J = \{J(t,dx) = K(t,x)\Gamma(t,dx),\ t \in [0,T]\}$ *is a predictable process with values in* $L^p(\Omega \times [0,T]; U \otimes H)$, *so* $G \cdot W$ *is well-defined. Furthermore,*

$$E\left(\|J \cdot W\|_H^p\right) \leqslant C \int_0^T dt \left(\sup_{x \in \mathbb{R}^d} E(\|K(t,x)\|_H^p) \right) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t)(\xi)|^2.$$

It becomes possible to integrate the Malliavin derivative of:
- the solution to the heat equation in any spatial dimension;
- the solution of the wave equation in spatial dimensions $k \in \{1,2,3\}$.

School B (K-R 1979, DP-Z 1992): Hilbert-space-valued integrals

Central notion: Hilbert-space-valued Brownian motion

Given: a separable Hilbert space V, a linear, symmetric (self-adjoint) non-negative definite and bounded operator Q on V such that $\text{Tr}\, Q < +\infty$ or $\text{Tr}\, Q = +\infty$.

Let $(e_j)_j$ be a c.o.n.b. of V that consists of eigenvectors of Q with eigenvalues λ_j, $j \in \mathbb{N}^*$.

Let $(\beta_j)_j$ be a sequence of i.i.d. real-valued standard B.M.'s. Define

$$\mathcal{W}_t = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \beta_j(t) e_j.$$

This is a V-valued Q-Wiener process if $\text{Tr}\, Q < +\infty$ and a "V-valued" cylindrical Q-Wiener process if $\text{Tr}\, Q = +\infty$.

Careful: if $\text{Tr}\, Q = +\infty$, then the series does not converge in V.

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Stochastic integrals with respect to $\mathcal{W}$

For $s \geqslant 0$, let $\Phi_s : V \to H$ be a (predictable) Hilbert-Schmidt operator. Consider stochastic integrals of the form

$$\int_0^T \Phi_t \, d\mathcal{W}_t.$$

Possible integrands: operators such that

$$E\Big(\int_0^T \|\Phi_t\|^2_{HS(Q^{1/2}(V),H)} \, dt\Big) < \infty.$$

Here,

$$\|\Phi_t\|^2_{HS(Q^{1/2}(V),H)} = \sum_{j=1}^{\infty} \|\Phi_t(\sqrt{\lambda_j}\, e_j)\|^2_H.$$

Series representation

Proposition

Let $(f_k)_k$ *be a c.o.n.b. of* H*. Assume that* $\Phi = \{\Phi_t,\ t \in [0, T]\}$ *is integrable. Then*

$$\int_0^T \Phi_t \, d\mathcal{W}_t = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \int_0^T \langle \Phi_t(\sqrt{\lambda_j}\, e_j), f_k \rangle_H \, d\beta_j(t) \right) f_k.$$

Remark. This formula is valid in both cases $\mathrm{Tr}\, Q < +\infty$ and $\mathrm{Tr}\, Q = +\infty$.

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Remark. This formula is valid in both cases Tr $Q < +\infty$ and Tr $Q = +\infty$.

Spatially homogeneous noise as a V-valued Wiener process

Take $V = U^*$, $Q = \mathrm{Id}_{U^*}$.

Let $W = \{W_t(\varphi),\ t \geqslant 0,\ \varphi \in U\}$ be the cylindrical B.M. associated with spatially homogeneous noise.

Define $\mathcal{W}_t^*$ to be the linear map $h \mapsto W_t(h)$, so that

$$\langle \mathcal{W}_t^*, h \rangle_{U^*,U} = W_t(h).$$

Careful: since $\mathrm{Tr}\ \mathrm{Id}_{U^*} = +\infty$, this process does not take values in U^*.

Fact: $(\mathcal{W}_t^*,\ t \geqslant 0)$ thus constructed is a cylindrical (standard) U^*-valued Wiener process.

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Relationship between Hilbert-space-valued integrals

(A) Walsh-Dalang-Sanz-Solé-Nualart-Quer-Sardanyons integral

(B) Krylov-Rozovskii, Da Prato-Zabczyk integral

In approach (A), one integrates tensor product-valued processes, with values in $V_Q \otimes H$ (recall that $V_Q \supset V$, $\langle h, g \rangle_{V_Q} = \langle Qh, g \rangle_V$).

In approach (B), one integrates Hilbert-Schmidt-valued processes, with values in $HS(Q^{1/2}(V), H)$.

Fact. $V_Q \otimes H \equiv HS(V_Q^*, H)$ and $V_Q^* = Q^{1/2}(V)$, so

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Equivalence of H-valued integrals

Proposition (D. & Quer-Sardanyons)

Let $\Phi = \{\Phi_t,\ t \in [0,T]\} \in L^2(\Omega \times [0,T]; HS(Q^{1/2}(V), H))$. *Let* $g^\Phi = \{g_t^\Phi,\ t \in [0,T]\} \in L^2(\Omega \times [0,T]; V_Q \otimes H)$ *be associated to* Φ *as follows:*

$$g_t^\Phi = \sum_{j,k=1}^{\infty} (g_t^\Phi)^{j,k} \Big(\frac{1}{\sqrt{\lambda_j}}\, e_j \otimes f_k\Big), \qquad \textit{with} \quad (g_s^\Phi)^{j,k} = \Big\langle \Phi_s(\sqrt{\lambda_j}\, e_j), f_k \Big\rangle_H. \quad (3)$$

Then

$$\int_0^T g_t^\Phi \, dW_t = \int_0^T \Phi_t^g \, d\mathcal{W}_t,$$

where the l.h.s. is a W-D-SS-N-Q-integral and the r.h.s. is a K-R-DP-Z-integral.

Stochastic p.d.e.'s: School A

Wave equation, spatial dimension $k = 3$:

A *mild random field* solution is a real-valued adapted stochastic process $\{u(t,x),\,(t,x) \in [0,T] \times \mathbb{R}^d\}$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))\, W(ds,dy) \qquad (4)$$

where the integral is a D-N-Q-integral (real-valued) and

$$I_0(t,x) = \frac{1}{4\pi t^2} \int_{\mathbb{R}^3} \left(t v_0(x-y) + u_0(x-y) + \nabla u_0(x-y) \cdot y\right)\, \sigma_t(dy).$$

Stochastic wave equation: School A

Theorem 2 (Dalang 1999, Nualart & Quer-Sardanyons 2007)

If $u_0 \in C^1(\mathbb{R}^3)$, u_0 and ∇u_0 are bounded, v_0 is bounded and continuous, and σ is Lipschitz. Suppose that the spectral measure μ of the noise satisfies

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < +\infty. \tag{5}$$

Then $\exists!$ mild random field solution u of (4). Moreover, the solution is L^2-continuous and for all $p \geqslant 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^p) < +\infty. \tag{6}$$

Remark: For each fixed (t,x), the solution is a real-valued random variable $u(t,x)$.

Stochastic p.d.e.'s: School B

Generic spde:

$$\begin{cases} du(t) = (Au(t) + F(u(t)))\,dt + B(u(t))\,d\mathcal{W}_t, & t \in]0, T], \\ u(0) = h. \end{cases} \tag{7}$$

$(\mathcal{W}_t)$: a cylindrical Q-Wiener process on a Hilbert space V,
$h \in H$, $F : H \to H$, and $B : H \to L(U, H)$,
$A : D(A) \subset H \to H$ is the infinitesimal generator of a (strongly) continuous semigroup $(S(t))_{t\in\mathbb{R}_+}$.

Generic solution: An adapted H-valued process $(u(t),\ t \in [0, T])$ is a *mild* solution of (7) provided for all $t \in [0, T]$, a.s.,

$$u(t) = S(t)h + \int_0^t S(t-s)F(u(s))\,ds + \int_0^t S(t-s)B(u(s))\,d\mathcal{W}_s. \tag{8}$$

Stochastic p.d.e.'s: School B

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The wave equation, spatial dimension $k = 3$

$$u(t) = \frac{\partial}{\partial t}\left(G(t) * u_0\right) + G(t) * v_0 + \int_0^t G(t-s) * \sigma(u(s))\, d\mathcal{W}_s^*. \qquad (9)$$

Choice of the value space H?
Interpretation of $G(t-s) * \sigma(u(s))$ as a HS-operator?

Set $H = L^2_\vartheta := L^2(\mathbb{R}^d, \vartheta(x)dx)$, where $\vartheta \in C^\infty(\mathbb{R}^d)$, $\vartheta > 0$, ϑ even and $\vartheta(x) = e^{-|x|}$, for $|x| \geqslant 1$.

Lemma (Peszat & Zabczyk, 2000)

If $u \in L^2_\vartheta$, then $G(s) * \sigma(u)$ can be interpreted as an element of $HS(U^*, L^2_\vartheta)$.

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Solution in the School B framework

Let H^1_ϑ be the weighted Sobolev space with the norm

$$\|\psi\|_{H^1_\vartheta} = \left(\int_{\mathbb{R}^d} \left[|\psi(x)|^2 + |\nabla \psi(x)|^2 \right] \vartheta(x) dx \right)^{1/2}.$$

Theorem 3 (Peszat & Zabczyk, 2000)

If $u_0 \in H^1_\vartheta$, $v_0 \in L^2_\vartheta$, σ is Lipschitz and the spectral measure satisfies

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < +\infty. \tag{10}$$

then $\exists!$ L^2_ϑ-valued solution to (9).

Remark: For each fixed t, the solution $u(t)$ is a (random) function of x, defined for a.e. x.

Relationship between solutions

In general, each approach starts with its own driving process, that has the covariance structure given by spatially homogeneous noise. So one cannot in general compare the solutions, but only their law.

However, here, both the martingale measure that appears in the School A version of the wave equation, and the cyl. Wiener process that appears in the School B version of the equation, are constructed from the same original spatially homogeneous noise.

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Equivalence of solutions

Theorem 4 (D. & Quer-Sardanyons)

For the stochastic heat equation in all spatial dimensions $k \geqslant 1$, or for the stochastic wave equation in spatial dimensions $k \in \{1, 2, 3\}$, let

$$(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k)$$

be the random field solution of the equation.

For each $t \in \mathbb{R}_+$, set $u(t) := u(t, *)$.

Then

$$(u(t),\ t \in \mathbb{R}_+)$$

is the L^2_ϑ-valued solution of the equation.

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Theorem 4 (D. & Quer-Sardanyons)

For the stochastic heat equation in all spatial dimensions $k \geqslant 1$, or for the stochastic wave equation in spatial dimensions $k \in \{1, 2, 3\}$, let

$$(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k)$$

be the random field solution of the equation.

For each $t \in \mathbb{R}_+$, set $u(t) := u(t,*)$.

Then

$$(u(t),\ t \in \mathbb{R}_+)$$

is the L^2_ϑ-valued solution of the equation.

Equivalence of solutions

Theorem 4 (D. & Quer-Sardanyons)

For the stochastic heat equation in all spatial dimensions $k \geqslant 1$, *or for the stochastic wave equation in spatial dimensions* $k \in \{1, 2, 3\}$, *let*

$$(u(t, x),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^k)$$

be the random field solution of the equation.

For each $t \in \mathbb{R}_+$, *set* $u(t) := u(t, *)$.

Then

$$(u(t),\ t \in \mathbb{R}_+)$$

is the L^2_ϑ*-valued solution of the equation.*

Conclusions

Many integrals that have been used over the last 30 years are closely related.

Our paper should help those who are using one integral and want to make use of results from a paper that uses an (a priori) different integral.

Main references

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