Stochastic heat equation with spatially-colored random forcing

Mohammud Foondun

School of Mathematics
Loughborough University

June 29, 2010

Outline

Outline

1. Intermittence and nonlinear parabolic stochastic partial differential equations, EJP(2009);joint with Davar Khoshnevisan
2. On the Stochastic heat equation with spatially-colored random forcing(preprint); joint with Davar Khoshnevisan

- Non-linear stochastic heat equation

$$
\begin{aligned}
\partial_t u(t\,,x) &= (\mathcal{L}u)(t\,,x) + \sigma(u(t\,,x))\dot{w}(t\,,x) \\
u(0\,,x) &= u_0(x)
\end{aligned}
$$

- Our aim: Existence & uniqueness and long time behavior

A special case

Upon taking $\mathcal{L} = \Delta$ and $\sigma(u) = u$, we end up with which is the called Parabolic anderson model:

$$\begin{aligned} \partial_t u(t\,x) &= (\Delta u)(t\,,x) + u(t\,,x)\dot{w}(t\,,x) \\ u(0\,,x) &= u_0(x) \end{aligned}$$

This is related to the stochastic Burgers equation and to the KPZ equation. See the work of Kardar and Zhang.

Previous work

- Carmona and Molchanov

$$\begin{aligned} \partial_t u(t\, x) &= (\Delta u)(t\,,x) + u(t\,,x)\xi(t\,,x) \quad x \in \mathbf{Z}^d \\ u(0\,,x) &= 1 \end{aligned}$$

- $u(t\,,x)$ is intermittent
- $u(t\,,x)$ consists of “high peaks and low valleys"

Mathematical definition of Intermittency

let

$$\lambda_p = \lim_{t\to\infty} \frac{\log \mathrm{E}[|u(t,\, x)|]^p}{t}$$

Hölder's inequality yields the following

$$\lambda_1 \leq \frac{\lambda_2}{2} \leq \frac{\lambda_3}{3} \leq \cdots$$

But if the following strict inequality is satisfied

$$\lambda_1 < \frac{\lambda_2}{2} < \frac{\lambda_3}{3} < \cdots ,$$

then the solution $u(t,\, x)$ is said to be intermittent.

The work of Bertini and Cancrini

- $$\begin{aligned} \partial_t u(t\, x) &= \frac{\nu}{2}(\Delta u)(t\,,x) + u(t\,,x)\dot{w}(t\,,x) \\ u(0\,,x) &= u_0(x) \end{aligned}$$
- They showed that the solution is intermittent.
- In fact they obtained

$$\lambda_p = \frac{1}{4!\nu} p(p^2 - 1)$$

The method

Their method consists of

- Smooth out the white noise
- Use Feynman-Kac representation which is in terms of local times
- Computed the moments
- Take limit

- $$\begin{aligned} \partial_{tt} u(t\,,x) &= \Delta u(t\,,x) + u(t\,,x)\dot{F}(t\,,x) \\ u(0\,,x) &= u_0, \quad \partial_t u(0\,,x) = \tilde{u}_0 \quad x \in \mathbf{R}^3. \end{aligned}$$

- $$\begin{aligned} \bar{\lambda}_n &:= \limsup_{t\to\infty} \frac{\log E[|u(t\,,x)|^n]}{t} \\ \underline{\lambda}_n &:= \liminf_{t\to\infty} \frac{\log E[|u(t\,,x)|^n]}{t} \end{aligned}$$

- Dalang and Mueller: $\frac{\bar{\lambda}_n}{n} \le c_1 n^{1/3}$ for all n and $\frac{\underline{\lambda}_n}{n} \ge c_2 n^{1/3}$ for even n.

Outline

- $\mathcal{L} :=$ L^2-generator of a Lévy process $\{X_t\}_{t\geq 0}$
- $\mathrm{E}\exp(i\xi X_t) = \exp(-t\Psi(\xi))$
- Formula for $\mathcal{L}$

$$\begin{aligned} \mathcal{L}f(x) &= -< a, f'(x) > + \frac{1}{2}\sum_{i,j=1,\cdots,d} Q_{ij} f''_{ij}(x) \\ &+ \int_{\mathbf{R}^d} (f(x+y) - f(x) - 1_{\{|y|<1\}} < y, f'(x) >)\nu(dy) \end{aligned}$$

- $\mathcal{L} := L^2-$generator of a Lévy process $\{X_t\}_{t\geq 0}$
- $\mathrm{E}\exp(i\xi X_t) = \exp(-t\Psi(\xi))$
- Formula for $\mathcal{L}$

$$\begin{aligned}\mathcal{L}f(x) &= -< a, f'(x) > + \frac{1}{2}\sum_{i,j=1,\cdots,d} Q_{ij} f''_{ij}(x) \\ &+ \int_{\mathbf{R}^d} (f(x+y) - f(x) - 1_{\{|y|<1\}} < y, f'(x) >)\nu(dy)\end{aligned}$$

- $\mathcal{L} := L^2-$generator of a Lévy process $\{X_t\}_{t\geq 0}$
- $\mathrm{E}\exp(i\xi X_t) = \exp(-t\Psi(\xi))$
- Formula for $\mathcal{L}$

$$\begin{aligned}\mathcal{L}f(x) &= -<a, f'(x)> + \frac{1}{2}\sum_{i,j=1,\cdots,d} Q_{ij} f''_{ij}(x) \\ &+ \int_{\mathbf{R}^d} (f(x+y) - f(x) - 1_{\{|y|<1\}} <y, f'(x)>)\nu(dy)\end{aligned}$$

local times

- Occupation measure: for Borel measurable $f : \mathbf{R}^d \longrightarrow \mathbf{R}$,

$$\int_{\mathbf{R}^d} f(x)\mu_t(dx) = \int_0^t f(X_s)ds.$$

- Local time

$$\mu_t(dx) = L(t,x)dx$$

-

$$L(t,x) = \lim_{\epsilon \longrightarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_s - x| < \epsilon\}} ds.$$

local times

- Occupation measure: for Borel measurable $f : \mathbf{R}^d \longrightarrow \mathbf{R}$,

$$\int_{\mathbf{R}^d} f(x)\mu_t(dx) = \int_0^t f(X_s)ds.$$

- Local time

$$\mu_t(dx) = L(t, x)dx$$

-

$$L(t, x) = \lim_{\epsilon \longrightarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_s - x| < \epsilon\}} ds.$$

local times

- Occupation measure: for Borel measurable $f : \mathbf{R}^d \longrightarrow \mathbf{R}$,

$$\int_{\mathbf{R}^d} f(x)\mu_t(dx) = \int_0^t f(X_s)ds.$$

- Local time

$$\mu_t(dx) = L(t,x)dx$$

- $$L(t,x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t 1_{\{|X_s - x|<\epsilon\}} ds.$$

Results with E. Nualart

- $$\begin{aligned} \partial_t u(t\, x) &= (\mathcal{L}u)(t\,,x) + \dot{w}(t\,,x) \\ u(0\,,x) &= u_0(x) \end{aligned}$$
- There exists a function valued solution iff $\bar{X} = X - X'$ has local time.

- Condition for function-valued solution

$$\begin{aligned}\Upsilon(\beta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\xi}{\beta + 2\mathrm{Re}\,\Psi(\xi)} \quad \beta > 0\\ &< \infty.\end{aligned}$$

- Hawkes condition

$$\int_{\mathbf{R}^d} \mathrm{Re}\left(\frac{1}{1+\Psi(\xi)}\right) d\xi < \infty.$$

- Hence solution exists iff $\bar{X} = X - X'$ has local time
- In fact $u(t\,,x)$ inherits a lot of properties from the local time of $\bar{X}$.

Our problem

- $$\begin{aligned} \partial_t u(t\, x) &= (\mathcal{L}u)(t\,,x) + \sigma(u(t\,,x))\dot{w}(t\,,x) \\ u(0\,,x) &= u_0(x) \end{aligned}$$

- $$\Upsilon(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2\mathrm{Re}\,\Psi(\xi)} < \infty$$

- Upper Liapunov exponent

$$\bar{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{E}(|u(t\,,x)|^p)$$

Our problem

- $$\begin{aligned}\partial_t u(t\,x) &= (\mathcal{L}u)(t\,,x) + \sigma(u(t\,,x))\dot{w}(t\,,x)\\ u(0\,,x) &= u_0(x)\end{aligned}$$

- $$\Upsilon(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2\mathrm{Re}\,\Psi(\xi)} < \infty$$

- Upper Liapunov exponent

$$\bar{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{E}(|u(t\,,x)|^p)$$

Our problem

- $$\begin{aligned} \partial_t u(t\,x) &= (\mathcal{L}u)(t\,,x) + \sigma(u(t\,,x))\dot{w}(t\,,x) \\ u(0\,,x) &= u_0(x) \end{aligned}$$

- $$\Upsilon(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2\mathrm{Re}\,\Psi(\xi)} < \infty$$

- Upper Liapunov exponent

$$\bar{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{E}(|u(t\,,x)|^p)$$

Existence results

- There exists a unique solution satisfying

$$\bar{\gamma}(p) \leq \inf\{\beta > 0 : \Upsilon\left(\frac{2\beta}{p}\right) < \frac{1}{(z_p \mathrm{Lip}_\sigma)^2}\} < \infty,$$

where z_p is the largest +ve zero of the Hermite polynomial He_p.

- Assume that $\inf_{z\in\mathbf{R}} u_0(z) > 0$ and $q := \inf_{x\neq 0} |\sigma(x)/x| > 0$ then $\bar{\gamma}(2) > \Upsilon^{-1}\left(\frac{1}{q^2}\right) > 0$,, where

$$\Upsilon^{-1}(t) : \sup\{\beta > 0 : \Upsilon(\beta) > t\}$$

Sharpness of the previous result

- If σ are bounded, then for all integers $p \geq 2$,

$$\mathrm{E}(|u(t\,,x)|^p) = o(t^{p/2}) \quad \text{as} \quad t \to \infty.$$

- If $\bar{X}$ is transient, then for all integers $p \geq 2$, there exists $\delta(p) > 0$ such that $\bar{\gamma}(p) = 0$ whenever $\mathrm{Lip}_\sigma < \delta(p)$
- Take $\Psi(\xi) = |\xi|^\alpha + |\xi|^\rho$. If $\alpha \in (0,1)$ and $\rho \in (1,2]$, then X is transcient and have local time.

Sharpness of the previous result

- If σ are bounded, then for all integers $p \geq 2$,

$$\mathrm{E}(|u(t\,,x)|^p) = o(t^{p/2}) \quad \text{as} \quad t \to \infty.$$

- If $\bar{X}$ is transient, then for all integers $p \geq 2$, there exists $\delta(p) > 0$ such that $\bar{\gamma}(p) = 0$ whenever $\mathrm{Lip}_\sigma < \delta(p)$
- Take $\Psi(\xi) = |\xi|^\alpha + |\xi|^\rho$. If $\alpha \in (0,1)$ and $\rho \in (1,2]$, then X is transcient and have local time.

Sharpness of the previous result

- If σ are bounded, then for all integers $p \geq 2$,

$$\mathrm{E}(|u(t\,,x)|^p) = o(t^{p/2}) \quad \text{as} \quad t \to \infty.$$

- If $\bar{X}$ is transient, then for all integers $p \geq 2$, there exists $\delta(p) > 0$ such that $\bar{\gamma}(p) = 0$ whenever $\mathrm{Lip}_\sigma < \delta(p)$
- Take $\Psi(\xi) = |\xi|^\alpha + |\xi|^\rho$. If $\alpha \in (0,1)$ and $\rho \in (1,2]$, then X is transcient and have local time.

A corollary

If $\sigma(x) := \lambda x$ and $\inf_{x\in\mathbf{R}} u_0(x) > 0$ then

- If $\bar{X}$ is transcient, then u is weakly intermittent if and only if $\Upsilon(\beta) \geq \lambda^{-2}$ for some $\beta > 0$
- If u is weakly intermittent, then $\bar{\gamma}(2) = \Upsilon^{-1}(\lambda^{-2})$

Here weakly intermittent is just Molchanov's definition but with upper Liapounov exponent instead of the Liapounov exponent.

A corollary

If $\sigma(x) := \lambda x$ and $\inf_{x\in\mathbf{R}} u_0(x) > 0$ then

- If $\bar{X}$ is transcient, then u is weakly intermittent if and only if $\Upsilon(\beta) \geq \lambda^{-2}$ for some $\beta > 0$
- If u is weakly intermittent, then $\bar{\gamma}(2) = \Upsilon^{-1}(\lambda^{-2})$

Here weakly intermittent is just Molchanov's definition but with upper Liapounov exponent instead of the Liapounov exponent.

- $$\Upsilon(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2\mathrm{Re}\,\Psi(\xi)} < \infty$$

- $$(\mathcal{G}u_0)(t\,,x) := \int_0^{\infty} p_t(y-x)u_0(y)\,dy$$

- $$u(t\,,x) = (\mathcal{G}u_0)(t\,,x) + \int_{-\infty}^{\infty}\int_0^t \sigma(u(s\,,y))p_{t-s}(y-x)\,w(ds\,dy).$$

- Set

$$(\mathcal{A}f)(t\,,x) := \int_{-\infty}^{\infty}\int_0^t \sigma(f(s\,,y))p_{t-s}(y-x)\,w(ds\,dy),$$

for all $t \geq 0$ and $x \in \mathbf{R}$

-

$$\|f\|_{p,\beta} := \left\{\sup_{t\geq 0}\sup_{x\in\mathbf{R}} e^{-\beta t}\mathrm{E}\left(|f(t\,,x)|^p\right)\right\}^{1/p}.$$

Upper bound

- Choose and fix an even integer $p \geq 2$. For every $\beta > 0$, and all predictable random fields f and g that satisfy $\|f\|_{p,\beta} + \|g\|_{p,\beta} < \infty$,

$$\|\mathcal{A}f - \mathcal{A}g\|_{p,\beta} \leq z_p \mathrm{Lip}_\sigma \sqrt{\Upsilon\left(\frac{2\beta}{p}\right)} \|f - g\|_{p,\beta}.$$

- Define $v_0(t,x) := u_0(x)$ and iteratively set

$$v_{n+1}(t\,,x) := (\mathcal{A}v_n)(t\,,x) + (\mathcal{G}u_0)(t\,,x) \qquad \text{for all } n \geq 0.$$

- Existence and upper bound on growth follows from the above.

We need to show that

$$\int_0^\infty e^{-\beta t}\mathrm{E}\left(|u(t\,,x)|^2\right)dt=\infty \quad \text{provided that } \Upsilon(\beta)\geq q^{-2}.$$

We will need the following notations:

$$\begin{aligned}
F_\beta(x) &:= \int_0^\infty e^{-\beta t}\mathrm{E}\left(|u(t\,,x)|^2\right)dt\\
G_\beta(x) &:= \int_0^\infty e^{-\beta t}\left|(\mathcal{G}u_0)(t\,,x)\right|^2 dt\\
H_\beta(x) &:= \int_0^\infty e^{-\beta t}\left|p_t(x)\right|^2 dt.
\end{aligned}$$

Using

$$u(t\,,x)=(\mathcal{G}u_0)(t\,,x)+\int_{-\infty}^{\infty}\int_0^t\sigma(u(s\,,y))p_{t-s}(y-x)\,w(ds\,dy).$$

, we obtain

$$\mathrm{E}\left(|u(t\,,x)|^2\right)$$

$$=|(\mathcal{G}u_0)(t\,,x)|^2+\int_{-\infty}^{\infty}dy\int_0^t ds\,\mathrm{E}\left(|\sigma(u(s\,,y))|^2\right)|p_{t-s}(y-x)|^2\,,$$

We then apply Laplace transform to obtain

$$F_\beta(x)=G_\beta(x)+\int_{-\infty}^{\infty}dy\,H_\beta(x-y)\int_0^{\infty}ds\,e^{-\beta s}\mathrm{E}\left(|\sigma(u(s\,,y))|^2\right).$$

lower bound

- Using $|\sigma(z)|^2 \geq q^2|z|^2$ for all $z \in \mathbf{R}$ we obtain

$$F_\beta(x) \geq G_\beta(x) + q^2(F_\beta * H_\beta)(x).$$

- Using the fact that $\eta := \inf_x u_0(x)$, we can iterate the above and end up with

$$F_\beta(x) \geq \eta^2\beta^{-1}\sum_{n=0}^{\infty}(q^2\Upsilon(\beta))^n,$$

- Hence $F_\beta(x) = \infty$ as long as $\Upsilon(\beta) \geq q^{-2}$.

The equation

- $$\begin{aligned} \partial_t u(t\, x) &= (\mathcal{L}u)(t\,,x) + \sigma(u(t\,,x))\dot{F}(t\,,x) \\ u(0\,,x) &= u_0(x) \end{aligned}$$
- $$\mathrm{E}[\dot{F}(s\,,x)\dot{F}(t\,,y)] = \delta_0(x-y)f(x-y)$$
- We assume that $\hat{f}$ is a measurable function.

- Condition: $\bar{R}_\alpha f(0) < \infty$ where

$$\bar{R}_\alpha f(x) := \int_0^\infty e^{-\alpha s} (\bar{P}_s f)(x)\, ds$$

- This is equivalent to the condition of Dalang:

$$\Upsilon_f(\beta) := \frac{1}{(2\pi)^d} \int \frac{\hat{f}(\xi)}{\beta + 2\mathrm{Re}\,\Psi(\xi)}\, d\xi < \infty$$

Existence result

- There exists a unique solution satisfying the following

$$\bar{\gamma}_*(p) \leq \inf\{\beta > 0 : Q(p\,,\beta) < 1\},$$

where $Q(p\,,\beta) := z_p Lip_\sigma \sqrt{\bar{R}_{2\beta/p} f(0)}$ and

$$\bar{\gamma}_*(p) := \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbf{R}^d} \ln \mathrm{E}(|u_t(x)|^p).$$

Suppose

- $\hat{f}(\xi)$ depends on $\xi \in \mathbf{R}^d$only through $|\xi_1|, \cdots, |\xi_d|$
- $|\xi_j| \mapsto \hat{f}(\xi)$ is nonincreasing for every $j = 1, \cdots, d$
- $\mathrm{Re}\,\Psi(\xi)$ depends on $\xi \in \mathbf{R}^d$ only through $|\xi_1|, \cdots, |\xi_d|$

- Ornstein-Uhlenbeck-type kernels

$$f(x) = c_1 e^{-c_2\|x\|^\alpha} \quad \hat{f}(\xi) = \frac{c_1}{(2\pi)^d} \int_{R^d} e^{-i\xi\cdot x - c_2\|x\|^\alpha}\, dx$$

- Riesz Kernels

$$f(x) = \frac{c}{\|x\|^\alpha} \quad \hat{f}(\xi) = \frac{c'}{\|\xi\|^{d-\alpha}}$$

- Further assume that $\eta := \inf_{x\in\mathbf{R}^d} u_0(x) > 0$ and there exists $L_\sigma \in (0,\infty)$ such that $\sigma(z) \geq L_\sigma|z|$ for all $z \in \mathbf{R}^d$, then

$$\inf_{x\in\mathbf{R}^d} \bar{\gamma}(2) \geq \sup\{\beta > 0 : (\bar{R}_\beta f)(0) \geq \frac{2^{d-1}}{L_\sigma^2}\}$$

- $$u(t\,,x)=(\mathcal{G}u_0)(t\,,x)+\int_{-\infty}^{\infty}\int_0^t\sigma(u(s\,,y))p_{t-s}(y-x)\,F(ds\,dy).$$
- $\inf_{x\in\mathbf{R}^d}\int_0^\infty e^{-\beta t}\mathrm{E}(|u(t\,,x)|^2)\,dt\geq$

$$\frac{\eta^2}{\beta}\sum_{l=0}^{\infty}\frac{L_\sigma^{2l}}{(2\pi)^{ld}}\int_{R^d}dz_1\cdots\int_{R^d}dz_d\prod_{j=1}^{l}\frac{\hat{f}(z_j-z_{j-1})}{\beta+2\mathrm{Re}\,\Psi(z_j)}$$

- $$\inf_{x\in\mathbf{R}^d}\int_0^\infty e^{-\beta}E(|u(t,x)|^2)dt=\infty \quad \Upsilon_f(\beta)\geq\frac{2^{d-1}}{L\sigma^2}$$