Rough viscosity solutions and applications to SPDEs

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Outline

I Rough differential equations (RDEs)
II Viscosity theory
III A (rough-) pathwise approach to SPDEs
IV BSDEs driven by rough paths

Motivation

- **T. Lyons ('98):** Let $(z^n) \subset C^1([0,T], \mathbb{R}^d)$ be Cauchy in *rough path metric*, with limit $\underline{\mathbf{z}}$. Assume

$$(\mathrm{ODE}): \ \dot{y}^n = V(y^n)\,\dot{z}^n, \ \ y^n(0) = y_0 \in \mathbb{R}^n,$$

where $V = (V_1, \ldots, V_d)$ are suitable vector fields.
Then y^n converges uniformly to $y = y(\underline{\mathbf{z}}) \in C([0,T], \mathbb{R}^n)$, independent of the approximating sequence.

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- **Lions–Souganidis ('03):** Let $(z^n) \subset C^1([0,T], \mathbb{R}^d)$ be uniformly convergent to some $z \in C([0,T], \mathbb{R}^d)$. Assume

$$(\mathrm{visc.PDE}) : \ (\partial_t - F)u^n = H(Du^n)\dot{z}^n, \ \ u^n(0,\cdot) = u_0,$$

  where $F = F(Du, D^2u)$, $H = (H_1, \ldots, H_d)$ are suitable.
  Then u^n converges uniformly to $u = u(z) \in \mathrm{BUC}([0,T], \mathbb{R}^n)$, independent of the approximating sequence.

Rough differential equations

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- **Interpretation:** y is the solution to a *rough differential equation*, driven by the *rough path* $\underline{\mathbf{z}}$ write

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- What are *rough path metrics* and what are *rough paths* ?
- First example, not applicable to Brownian motion: take

$$\rho_{\alpha\text{-Höl}}(z, \tilde{z}) := \sup_{s,t \in [0,T]} \frac{|z_{s,t} - \tilde{z}_{s,t}|}{|t-s|^{\alpha}} \;\; \text{for } \alpha \in (1/2, 1];$$

  rough paths are just α-Hölder paths; RDEs are "Young" ODEs.

Rough differential equations (cont'd)

- A better example, applicable to Brownian motion, for $\alpha \in (1/3, 1/2]$ take

$$\rho_{\alpha\text{-Höl}}(z, \tilde{z}) := \sup_{s,t \in [0,T]} \frac{\left|\mathbf{z}^1_{s,t} - \tilde{\mathbf{z}}^1_{s,t}\right|}{|t-s|^{\alpha}} + \frac{\left|\mathbf{z}^2_{s,t} - \tilde{\mathbf{z}}^2_{s,t}\right|}{|t-s|^{2\alpha}}$$

where we introduced generalized increments of $z \in C^1$,

$$\mathbf{z}_{s,t} := \left(\mathbf{z}^1_{s,t}, \mathbf{z}^2_{s,t}\right) := \left(\int_s^t dz, \int_s^t \int_s^r dz \otimes dz\right) \in \mathbb{R}^d \otimes \mathbb{R}^{d \times d}.$$

The (abstract) completion of C^1-paths with respect to $\rho_{\alpha\text{-Höl}}$ leads to *rough path space* which can be identified as a subset of

$$\left\{\mathbf{z} \in C\left([0,T], \mathbb{R}^d \otimes \mathbb{R}^{d \times d}\right) : \sup_{s,t \in [0,T]} \frac{\left|\mathbf{z}^1_{s,t}\right|}{|t-s|^{\alpha}} + \frac{\left|\mathbf{z}^2_{s,t}\right|}{|t-s|^{2\alpha}} < \infty\right\}.$$

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- From $d\left(z^i z^j\right) = z^i dz^j + z^j dz^i \implies \mathrm{Sym}\left(\mathbf{z}^2\right) = \frac{1}{2}\mathbf{z}^1 \otimes \mathbf{z}^1 \implies$

$$\left(\mathbf{z}^1_{s,t}, \mathbf{z}^2_{s,t}\right) \leftrightarrow \left(\mathbf{z}^1_{s,t}, \mathbf{A}_{s,t}\right) \quad \text{with "area" } \mathbf{A}_{s,t} := \mathrm{Anti}\left(\mathbf{z}^2_{s,t}\right)$$

- **Example: Homogenization of highly oscillatory ODEs**

$$t \mapsto \underline{\mathbf{z}}_t \equiv \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \right)$$

is the rough path limit, any $\alpha \in (1/3, 1/2)$, of

$$z^n(t) = n^{-1} \exp\left(2\pi i n^2 t\right) \in \mathbb{C} \cong \mathbb{R}^2.$$

Given two vector fields $V = (V_1, V_2)$ the RDE solution

$$dy = V(y)\, d\underline{\mathbf{z}} \tag{1}$$

models the effective behaviour of the highly oscillatory ODE

$$dy^n = V(y^n)\, dz^n \quad \text{as } n \to \infty.$$

In fact, the RDE solution of (1) solves the ODE

$$\dot{y} = [V_1, V_2](y)$$

where $[V_1, V_2]$ is the Lie bracket of V_1 and V_2.

- **Stochastic differential equations**. Let B be d-dimensional Brownian motion. Since $B(\omega) \notin C^1$ careful interpretation of the *stochastic* differential equation

$$dy = V(y)\,\partial B$$

is necessary (Itô-theory). On the other hand we can set

$$\underline{\mathbf{B}}_t(\omega) = \left(B_t, \int_0^t B_s \otimes \partial B_s \right)$$

where ∂ indicates (Stratonovich) integration. For any $\alpha \in (1/3, 1/2)$

$$\mathbb{P}\left[\underline{\mathbf{B}} \text{ is a } \alpha\text{-Hölder rough path}\right] = 1.$$

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- Any reasonable (smooth) approximation to Brownian motion converges to $\underline{\mathbf{B}}$ in rough path metric.
- But: there are "unreasonable" approximations, e.g. those of **[McShane '72]**, which converge to

$$\underline{\mathbf{B}}_t + \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \right)$$

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- RDE solution to $dy = V(y)\, d\underline{\mathbf{B}}$ is solved for fixed ω, depends continuously on $\underline{\mathbf{B}}$ and yields a (classical) Stratonovich SDE solution.

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- $+$) consider solution flow ψ to

$$\text{(RDE)}: \; dy = V(y)\, d\underline{\mathbf{z}};$$

For e.g. $V \in Lip^{3+\varepsilon}$, can see that $\psi, D\psi, D^2\psi$ exist and depend continuously on $\underline{\mathbf{z}}$; also for $\psi^{-1}, D\psi^{-1}, D^2\psi^{-1}$. Limit theorems for stochastic flows as trivial consequence.

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- +) various applications of continuity in $\underline{\mathbf{B}}$; e.g. support theorems
- -) one additional degree of smoothness compared to Itô-theory.

Part II: Second order parabolic viscosity theory

- **Crandall, Evans, Ishii, Lions ('80ties)** ... : Let $u = u(t, x)$ and $Fu = F(t, x, u, Du, D^2 u)$ be continuous, weakly elliptic. Subject to (TC), satisfied in many examples (in particular from *stochastic control theory*)

$$(\partial_t - F)\, u = 0, \quad u(0, \cdot) = u_0 \in \mathrm{BUC}(\mathbb{R}^n)$$

has a unique solution, in viscosity sense, say $u \in \mathrm{BUC}([0, T], \mathbb{R}^n)$. In fact, one has comparison in the sense that

$$\begin{aligned} (\partial_t - F)\, u \quad &\leq \quad 0 \leq (\partial_t - F)\, v \\ \text{and} \quad u_0 \quad &\leq \quad v_0 \qquad \text{implies} \quad u \leq v. \end{aligned}$$

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- Here $(\partial_t - F)\, u \leq 0$ means $(\partial_t - F)\, \varphi|_{\bar{t},\bar{x}} \leq 0$ for any smooth test-function which touches u from above at $(\bar{t}, \bar{x})$.
- Motivation for this definition: if $u \in C^{1,2}$ is a classical subsolution,

$$\partial_t \varphi - F(t, x, \varphi, D\varphi, D^2\varphi) \leq \partial_t u - F(t, x, Du, D^2u) \leq 0.$$

Stability properties of viscosity theory

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- Nice feature: no need to show convergence of the derivatives
- **Stability [Barles–Perthame '88]** If $(\partial_t - F_\varepsilon)\, u_\varepsilon \leq 0$ with $F_\varepsilon \to F$ locally uniformly and $(\varepsilon, t, x) \mapsto u_\varepsilon(t,x)$ locally bounded; then

$$(\partial_t - F)\, \bar{u} \leq 0$$

where

$$\bar{u}(t,x) := \overline{\lim_{\varepsilon \to 0}}^{*} u_\varepsilon(t,x) = \overline{\lim_{\varepsilon \to 0, t' \to t, x' \to t}} u_\varepsilon(t', x').$$

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- *Typical application:* $(\partial_t - F_\varepsilon) u_\varepsilon = 0$ with $F_\varepsilon \to F$ locally uniformly; $\{u_\varepsilon\}$ locally bounded and comparison holds for $\partial_t - F$. Then u_ε converges local uniformly to some $u = u(t, x)$ and

$$(\partial_t - F) u = 0.$$

Part III: Stochastic viscosity solutions

- **Aim:** a pathwise theory of fully nonlinear SPDE of the form

$$\begin{aligned} du - F\left(t, x, u, Du, D^2 u\right) dt &= H\left(x, u, Du\right) \partial B, \\ u\left(0, \cdot\right) &= u_0 \in \mathrm{BUC}\left(\mathbb{R}^n\right) \end{aligned}$$

for $H = \left(H_1, \ldots, H_d\right)$ and a d-dimensional Brownian motion B.

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  for $H = (H_1, \ldots, H_d)$ and a d-dimensional Brownian motion B.
- **Theorem [Lions–Souganidis '03]** Fix $u_0 \in \mathrm{BUC}\left(\mathbb{R}^n\right)$ and let $z^n \in C^1\left([0, T], \mathbb{R}^d\right)$ uniformly convergent to some $z \in C\left([0, T], \mathbb{R}^d\right)$. Assume

$$\left(\partial_t - F\right) u^n = H\left(Du^n\right) \dot{z}^n, \quad u^n\left(0, \cdot\right) = u_0.$$

  Then u^n converges locally uniformly to a limit $u = u\left(z\right) \in \mathrm{BUC}\left([0, T], \mathbb{R}^n\right)$, independent of the approximating sequence. Interpretation:

$$du - F\left(t, x, Du, D^2 u\right) dt = H\left(Du\right) \partial z;$$

  applicable to a.e. (continuous) sample path of Brownian motion.

Remarks:

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- The case when $H = H\left(x, u\right)$ driven by a multi-dimensional Brownian motion is difficult.
- They also have a note [CRAS '00] on

$$du - F\left(Du, D^2 u\right) dt = H\left(u\right)\partial z.$$

Stability of solutions as functions of z only in rough path sense (unless $\left[H_i, H_j\right] = 0$); we shall return to this later. In this case, x-dependence does not add real difficulty.

- **Theorem [Caruana-F-Oberhauser AIHP '10]** Fix $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$ and let $z^n \in C^1([0,T], \mathbb{R}^d)$ Cauchy in rough path metric with (geometric) rough path limit $\underline{\mathbf{z}}$. Assume

$$(\partial_t - F)\, u^n = \langle Du^n, \sigma(x) \rangle\, \dot{z}^n, \quad u^n(0,\cdot) = u_0.$$

Then $u^n \to u = u(\underline{\mathbf{z}}) \in \mathrm{BUC}([0,T], \mathbb{R}^n)$ locally uniformly, independent of the approximating sequence. Interpretation:

$$du - F\left(t, x, Du, D^2 u\right) dt = \langle Du, \sigma(x) \rangle\, d\underline{\mathbf{z}};$$

applicable to a.e. sample path of $\underline{\mathbf{B}}$ i.e. Brownian motion and Lévy area.

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- **Remark:** with x-dependence a pathwise SPDE theory must be *rough* pathwise. Indeed, take $F = 0$, $\sigma = (V_1, ..., V_d)$. The resulting (linear, first-order) SPDE is solved globally by the method of characteristics and involves the flow of the SDE $dy = V(y) \partial B$. Now think of McShane type approximations to B.

- **Idea of proof:** we have $H(x, Du) = \langle Du, \sigma(x) \rangle$, linear in the gradient. Then

$$" dy^n = \sigma(y^n) dz^n \qquad \longrightarrow \quad dy = \sigma(y) d\underline{\mathbf{z}} \ "$$

as flow (of diffeomorphisms), say $\psi_n \to \psi$. This induces a new coordinate chart in which, setting $v^n(t,x) = u^n(t, \psi_n(t,x))$,

$$(\partial_t - F) u^n = \langle Du^n, \sigma(x) \rangle \dot{z}^n \Leftrightarrow (\partial_t - F^n) v^n = 0.$$

and where $F^n \equiv F^{\psi_n} \longrightarrow F^{\mathbf{z}} \equiv F^{\psi}$ locally uniformly. Assuming local boundedness of v^n then implies, using stability of viscosity solutions,

$$(\partial_t - F^{\mathbf{z}}) \left[\overline{\lim_{n\to\infty}}^* v^n(t,x) \right] \leq 0 \leq (\partial_t - F^{\mathbf{z}}) \left[\underline{\lim}^*_{n\to\infty} v^n(t,x) \right]$$

Comparison implies $v^n \to v$ locally uniformly and $(\partial_t - F^{\mathbf{z}}) v = 0$.

- Unwrapping the change-of-coordinates then yields a solution to the fully non-linear PDE with "rough path" noise

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- **Moral: stability properties of viscosity - and rough path theory work perfectly together.**
- Applications to stochastic PDEs are *rough*pathwise i.e. by taking

$$\underline{\mathbf{z}}_t = \underline{\mathbf{B}}_t\left(\omega\right) = \left(B_t, \int_0^t B_s \otimes \partial B_s\right)$$

obtain Stratonovich SPDE solution (no need for stochastic parabolicity assumption!)

- **Theorem [F-Oberhauser ArXiv '10]** Let $L = L\left(t, x, u, Du, D^2 u\right)$ and $M_i\left(t, x, u, Du\right)$ be *linear* differential operators. Let $u_0 \in BUC$ and $\mathbf{z}$ a rough path. Then there exists a unique $u = u^{\underline{\mathbf{z}}} \in BUC\left(\left[0, T\right] \times \mathbb{R}^n\right)$, write

$$du - L\left(t, x, u, Du, D^2 u\right) dt = M\left(t, x, u, Du\right) d\underline{\mathbf{z}}, \;\; u\left(0, \cdot\right) \equiv u_0,$$

such that for any smooth sequence $z^n \longrightarrow \underline{\mathbf{z}}$ in rough path metric, u^n, BUC viscosity solutions of

$$\dot{u}^n - L\left(t, x, u, Du^n, D^2 u^n\right) = M\left(t, x, u^n, Du^n\right) \dot{z}^n, \;\; u\left(0, \cdot\right) \equiv u_0,$$

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- **Useful?** Well-known SPDE results (support theorems, approximations theorems, splitting methods, maximum principles) become immediate corollaries.

- **Idea of proof:** W.l.o.g. noise term writtes as

$$\sum_{i=1}^{d} M_i\left(x, u, Du\right) dz^i = \sum_{j=1}^{d_1} \left\langle \sigma_j\left(x\right), Du\right\rangle d\xi^j + u \sum_{k=1}^{d_2} \eta_k\left(x\right) d\theta^k$$

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- **Inner transformation:** $\psi(t,\cdot)$ solution flow to $\dot{\psi} = \sigma(\psi)\,d\xi$
$\Longrightarrow\ w = u(t,\psi(t,x))$ solves

$$\left(\partial_t - L^{\psi}\right) w = w \sum_{k=1}^{d_1} \eta_k(\psi(t,x))\, d\theta^k =: w\,\eta(\psi)\, d\theta$$

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- **Outer transformation:** $\phi\left(t, \cdot; x\right)$ solution flow to $\dot{\phi} = \phi \nu\left(\psi\right) d\theta$ $\Longrightarrow$ $\log \phi$ is rough integral based on $\left(\psi, \theta\right)$. Then $v = \phi\left(t, w\left(t, x\right); x\right)$ solves

$$\left(\partial_t - {}^{\phi}\left[L^{\psi}\right]\right) v = 0$$

where ${}^{\phi}\left[L^{\psi}\right] = {}^{\phi}\left[L^{\psi}\right]\left(t, x, v, Dv, D^2 v\right)$ is linear if L is ...

What about non-linear noise?

- [Gubinelli–Tindel, AoP '10; Deya et al.] consider **non-linear rough heat equations**

$$du - (\Delta u)\, dt = H(u, x)\, d\underline{\mathbf{z}} \tag{2}$$

and establish uniqueness and local existence in Sobolev-spaces.

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and establish uniqueness and local existence in Sobolev-spaces.
- **In the present framework:** focus on $H = H(u)$ and Δ for simplicity. Perform **inner transformation** via solution flow of RDE $\dot{\phi} = H(\phi)\, d\underline{\mathbf{z}}$. Then $v = \phi^{-1}(t, u(t, x))$ solves
$$\partial_t v - (\Delta v)\, dt = \frac{\phi''(t, v)}{\phi'(t, v)} |Dv|^2 \equiv \alpha(t, v) |Dv|^2 .$$
Since $|\alpha(t, v)| < \varepsilon$ for $0 \leq t < h(\varepsilon)$ we can apply the comparison results of **[Kobylanski AoP '00 or Lions-Souganidis '00]** on small intervals. We then obtain, as before, solutions to (2) on $[0, h]$, then on $[h, 2h]$ ... and hence on $[0, T]$. Moreover, $(u_0, \underline{\mathbf{z}}) \mapsto u^{\underline{\mathbf{z}}}$ is continuous.

Part IV: BSDEs driven by rough paths

- Rewrite non-linear rough heat equation as terminal value problem

$$du + (Lu)\, dt = H(u, x)\, dz; \quad u_T = g$$

when $z \in C^1$ this relates to the BSDE **[Pardoux–Peng, '92]**

$$-dY_t^{x,s} = H\left(Y_t^{x,s}, X_t^{x,s}\right) dz_t - Z_t^{x,s} dB_t, \quad Y_T = g\left(X_T\right).$$

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- Work in progress (Diehl-F): consider

$$-dY_t = H\left(Y_t, \omega\right) dz_t - Z_t\left(\omega\right) dB_t, \quad Y_T = \xi \in L^\infty;$$

and write $\left(Y^n, Z^n\right)$ for the BSDE solution pair when $z = z^n$. Using stability results for "quadratic BSDEs" we can show that $z^n \to \underline{\mathbf{z}}$ entails convergence of $\left(Y^n, Z^n\right)$; giving meaning to

$$\text{(rough BSDE):} \; -dY_t = H\left(Y_t, \omega\right) d\underline{\mathbf{z}}_t - Z\left(\omega\right) dB_t,$$

Thank you for your attention!
References: just google: arxiv friz