Finite element approximation of the Cahn-Hilliard-Cook equation

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Outline

The Cahn-Hilliard-Cook equation:

$$
\begin{cases}
\mathrm{d}u - \Delta v \,\mathrm{d}t = \mathrm{d}W & \text{in } \mathcal{D} \times [0, T] \\
v + \Delta u + f(u) = 0 & \text{in } \mathcal{D} \times [0, T] \\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D} \times [0, T] \\
u(0) = u_0 & \text{in } \mathcal{D}
\end{cases}
$$

$$f(s) = s^3 - s, \quad \mathcal{D} \subset \mathbf{R}^d, \ d \leq 3$$

- Abstract framework
- Finite element approximation
- Linear CHC: strong convergence
- Linear CHC: weak convergence
- Nonlinear CHC: strong convergence

Co-workers

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CHC: abstract framework

$$\begin{cases} \mathrm{d}u - \Delta v\,\mathrm{d}t = \mathrm{d}W & \text{in } \mathcal{D} \times [0, T] \\ v + \Delta u + f(u) = 0 & \text{in } \mathcal{D} \times [0, T] \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D} \times [0, T] \\ u(0) = u_0 & \text{in } \mathcal{D} \end{cases}$$

Eliminate v.

Set $X = u \in H = L_2(\mathcal{D})$.

Let $A = -\Delta$ be the Neumann Laplacian in H.

W(t) – a Q-Wiener process on H with respect to $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$

$$\begin{cases} \mathrm{d}X + \big(A^2 X + Af(X)\big)\,\mathrm{d}t = \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

The Cahn-Hilliard semigroup

$A = -\Delta$ is the Neumann Laplacian in H.

Eigenvalues: $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to \infty.$

Orthonormal eigenbasis: $\{\varphi_j\}_{j=0}^{\infty}, \quad \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$

$$E(t)v = \mathrm{e}^{-tA^2} v = \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j + \langle v, \varphi_0 \rangle \varphi_0$$

$$\begin{cases} \dot{u} + A^2 u = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \Rightarrow u(t) = E(t)v$$

$$\begin{cases} \dot{u} + A^2 u = f, & t > 0 \\ u(0) = v \end{cases} \qquad \Rightarrow u(t) = E(t)v + \int_0^t E(t-s) f(s)\,\mathrm{d}s$$

CHC: abstract formulation

$$\begin{cases} \mathrm{d}X + A^2 X\,\mathrm{d}t = -Af(X)\,\mathrm{d}t + \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

The mild solution is given by the equation:

$$\begin{aligned} X(t) &= \mathrm{e}^{-tA^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s) \\ &= Y(t) + W_A(t) \end{aligned}$$

Stochastic convolution:

$$W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)$$

Random evolution problem:

$$\begin{cases} \dot{Y} + A^2 Y + Af(Y + W_A) = 0, & t > 0 \\ Y(0) = X_0. \end{cases}$$

The finite element method

Spatial discretization

- family of triangulations of $\mathcal{D}$: $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces: $\{S_h\}_{0<h<1}$
- $S_h \subset H^1(\mathcal{D})$, continuous piecewise linear functions
- $\begin{cases} \langle \mathrm{d}u_h, \chi\rangle + \langle \nabla v_h, \nabla\chi\rangle \,\mathrm{d}t = \langle \mathrm{d}W, \chi\rangle & \forall \chi \in S_h,\ t>0 \\ \langle v_h, \chi\rangle = \langle \nabla u_h, \nabla\chi\rangle + \langle f(u_h), \chi\rangle & \forall \chi \in S_h,\ t>0 \end{cases}$
- $A_h : S_h \to S_h,$ discrete Laplacian, $\quad \langle A_h\psi, \chi\rangle = \langle \nabla\psi, \nabla\chi\rangle,\ \forall \chi \in S_h$
- $P_h : H \to S_h,$ orthogonal projector, $\quad \langle P_h f, \chi\rangle = \langle f, \chi\rangle,\ \forall \chi \in S_h$
- $\begin{cases} \mathrm{d}X_h + A_h^2 X_h \,\mathrm{d}t + A_h P_h f(X_h)\,\mathrm{d}t = P_h\,\mathrm{d}W, & t>0 \\ X(0) = P_h X_0 \end{cases}$
- eigenvalues: $0 = \lambda_{h,0} < \lambda_{h,1} \le \cdots \le \lambda_{h,j} \le \cdots \le \lambda_{h,N_h}$
- orthonormal eigenbasis: $\{\varphi_{h,j}\}_{j=0}^{N_h}, \quad \varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$
- semigroup:

$$E_h(t)v_h = \mathrm{e}^{-tA_h^2} v_h = \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}^2} \langle v_h, \varphi_{h,j}\rangle \varphi_{h,j}$$

CHC: finite element approximation

$$\begin{cases} \mathrm{d}X_h + A_h^2 X_h \,\mathrm{d}t = -A_h P_h f(X_h)\,\mathrm{d}t + P_h \,\mathrm{d}W, \quad t > 0 \\ X(0) = P_h X_0 \end{cases}$$

The mild solution is given by the equation:

$$\begin{aligned} X_h(t) &= \mathrm{e}^{-tA_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X_h(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \,\mathrm{d}W(s) \\ &= Y_h(t) + W_{A_h}(t) \end{aligned}$$

Stochastic convolution: $W_{A_h}(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \,\mathrm{d}W(s)$

Random evolution problem:

$$\begin{cases} \dot{Y}_h + A_h^2 Y_h + A_h P_h f(Y_h + W_{A_h}) = 0, \quad t > 0; \\ Y_h(0) = P_h X_0. \end{cases}$$

Linear CHC: regularity

$$\begin{cases} \mathrm{d}X + A^2 X\,\mathrm{d}t = \mathrm{d}W, \quad t > 0 \\ X(0) = 0 \end{cases}$$

Stochastic convolution: $X(t) = W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)$

Seminorms: $|v|_\beta = \|A^{\beta/2}v\| = \Big(\sum_{j=1}^\infty \lambda_j^{\beta/2}\langle v,\varphi_j\rangle^2\Big)^{\frac{1}{2}}, \quad \dot{H}^\beta = D(A^{\beta/2}), \quad \beta \in \mathbf{R}$

Mean square norms: $\|v\|^2_{L_2(\Omega,\dot{H}^\beta)} = \mathbf{E}(|v|^2_\beta) = \int_\Omega \int_{\mathcal{D}} |A^{\beta/2}v|^2\,\mathrm{d}\xi\,\mathrm{d}\mathbf{P}(\omega), \quad \beta \in \mathbf{R}$

Theorem

If $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS} < \infty$ for some $\beta \geq 0$, then

$$\|X(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS}, \quad t \geq 0.$$

Linear CHC: strong convergence

$$H = L_2(\mathcal{D}), \quad \dot{H} = \left\{ v \in H : \int_{\mathcal{D}} v \, \mathrm{d}x = 0 \right\}$$

Orthogonal projector: $P \colon H \to \dot{H}$

$$W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2} \, \mathrm{d}W(s) = \int_0^t \mathrm{e}^{-(t-s)A^2} P \, \mathrm{d}W(s) + (I - P)W(t)$$

$$\begin{aligned} W_{A_h}(t) &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \, \mathrm{d}W(s) \\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h P \, \mathrm{d}W(s) + (I - P)W(t) \end{aligned}$$

$$\begin{aligned} W_{A_h}(t) - W_A(t) &= \int_0^t \left(\mathrm{e}^{-(t-s)A_h^2} P_h - \mathrm{e}^{-(t-s)A^2} \right) P \, \mathrm{d}W(s) \\ &= \int_0^t F_h(t-s) P \, \mathrm{d}W(s) \end{aligned}$$

Linear CHC: approximation of the semigroup

$$\begin{cases} \dot u + A^2 u = 0, \quad t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} \dot u_h + A_h^2 u_h = 0, \quad t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad\qquad u_h(t) = E_h(t)P_h v$$

Error: $F_h(t)v = E_h(t)P_h v - E(t)v$, seminorm: $|v|_\beta = \|A^{\beta/2}v\|$

Theorem

- $\|F_h(t)Pv\| \le Ch^\beta |v|_\beta, \quad t \ge 0, \quad \beta \in [0,2]$
- $\Big(\int_0^t \|F_h(s)Pv\|^2 \, ds \Big)^{1/2} \le Ch^\beta |\log(h)| \, |v|_{\beta-2}, \quad t \ge 0, \quad \beta \in [1,2]$

Linear CHC: strong convergence

Theorem

If $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS} < \infty$ *for some* $\beta \in [1,2]$*, then*

$$\|W_{A_h}(t) - W_A(t)\|_{L_2(\Omega,H)} \le Ch^{\beta}|\log(h)|\,\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS}, \quad t \ge 0$$

Proof.

$$\begin{aligned}
&\|W_{A_h}(t) - W_A(t)\|^2_{L_2(\Omega,H)} \\
&\quad = \mathbf{E}\Big\|\int_0^t F_h(t-s)P\,\mathrm{d}W(s)\Big\|^2 = \int_0^t \|F_h(t-s)PQ^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&\quad = \sum_{j=1}^{\infty}\int_0^t \|F_h(t-s)PQ^{1/2}\phi_j\|^2\,\mathrm{d}s \le C\sum_{j=1}^{\infty} h^{2\beta}|\log(h)|^2|Q^{1/2}\phi_j|^2_{\beta-2} \\
&\quad = Ch^{2\beta}|\log(h)|^2\sum_{j=1}^{\infty}\|A^{(\beta-2)/2}Q^{1/2}\phi_j\|^2 = Ch^{2\beta}|\log(h)|^2\|A^{(\beta-2)/2}Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}$$

□

Linear CHC: strong convergence

Larsson and Mesforush, IMAJNA (2010), to appear.
Euler timestepping is also studied here.

Heat equation:

- Printems (2001) (timestepping only)
- Yan (2004), (2005)

Wave equation:

- Kovács, Larsson, and Saedpanah, SINUM (2010)
- Quer-Sardanyons and Sanz-Solé (2006)
- Walsh (2006)

Strong and weak error

- Strong error:

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} = (\mathbf{E}\|X_h(t) - X(t)\|_H^2)^{1/2}$$

- Weak error:

$$\mathbf{E}\Big(G(X_h(T))\Big) - \mathbf{E}\Big(G(X(T))\Big)$$

for some class of functions $G : H \to \mathbb{R}$.

Weak error representation: preliminaries

Consider more generally:

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t),\ t \in (0,T]; \quad X(0) = X_0,$$

with solution $X(t) = E(t)X_0 + \int_0^t E(t-s)B\,\mathrm{d}W(s)$.

Apply integrating factor:

$$\mathrm{d}Y(t) = E(T-t)B\,\mathrm{d}W(t),\ t \in (0,T]; \quad Y(0) = E(T)X_0,$$

with solution $Y(t) = E(T)X_0 + \int_0^t E(T-s)B\,\mathrm{d}W(s)$.

Finite element approximations:

$$X_h(t) = E_h(t)X_{0,h} + \int_0^t E_h(t-s)B_h\,\mathrm{d}W(s).$$

$$Y_h(t) = E_h(T)X_{0,h} + \int_0^t E_h(T-s)B_h\,\mathrm{d}W(s).$$

Note: $X(T) = Y(T)$, $X_h(T) = Y_h(T)$. No drift term in eq. for Y, Y_h.

Weak error representation: preliminaries

Auxiliary problem:

$$\mathrm{d}Z(t) = E(T-t)B\,\mathrm{d}W(t),\ \ t \in (\tau, T];\ \ Z(\tau) = \xi,$$

where ξ is a $\mathcal{F}_\tau$-measurable random variable.

Unique solution: $Z(t,\tau,\xi) = \xi + \int_\tau^t E(T-s)B\,\mathrm{d}W(s)$

Define $u\colon H \times [0,T] \to \mathbb{R}$ by $u(x,t) = \mathbf{E}\Big(G(Z(T,t,x))\Big)$.

If $G \in C_{\mathrm{b}}^2(H,\mathbb{R})$, then u is a solution to Kolmogorov's equation:

$$\begin{cases} u_t'(x,t) + \frac{1}{2}\operatorname{Tr}\Big(u_{xx}''(x,t)E(T-t)BQB^*E(T-t)^*\Big) = 0,\ \ t \in [0,T),\ \ x \in H, \\ u(x,T) = G(x), \end{cases}$$

Weak error representation

Theorem

If $\operatorname{Tr}\Big(\int_0^T E(t)BQ[E(t)B]^*\,dt\Big) < \infty$ *and* $G \in C_{\mathrm{b}}^2(H,\mathbb{R})$, *then the weak error has the representation*

$$\begin{aligned}
&\mathbf{E}\Big(G(X_h(T))\Big) - \mathbf{E}\Big(G(X(T))\Big)\\
&\quad = \mathbf{E}\Big(u(Y_h(0),0) - u(Y(0),0)\Big)\\
&\qquad + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\Big(u''_{xx}(Y_h(t),t)\\
&\qquad\quad \times [E_h(T-t)B_h - E(T-t)B]Q[E_h(T-t)B_h + E(T-t)B]^*\Big)\,dt.
\end{aligned}$$

Proof: apply Itô's formula to $u(Y_h(t),t)$, use Kolomogorov's equation, manipulations with trace.

Linear CHC: weak convergence

$$X(t) = W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)$$

$$X_h(t) = W_{A_h}(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\,\mathrm{d}W(s)$$

$$\begin{aligned} &\mathbf{E}\Big(G(X_h(T))\Big) - \mathbf{E}\Big(G(X(T))\Big) \\ &\qquad = \frac{1}{2}\mathbf{E}\int_0^T \mathrm{Tr}\,\Big(u''_{xx}(Y_h(t),t) \\ &\qquad\quad \times [\mathrm{e}^{-(T-t)A_h^2}P_h - \mathrm{e}^{-(T-t)A^2}]Q[\mathrm{e}^{-(T-t)A_h^2}P_h + \mathrm{e}^{-(T-t)A^2}]^*\Big)\,\mathrm{d}t \end{aligned}$$

Approximation of the semigroup: $F_h(t) = \mathrm{e}^{-(T-t)A_h^2}P_h - \mathrm{e}^{-(T-t)A^2}$

$$\|F_h(t)v\| \le Ch^s|v|_s, \quad t \ge 0, \quad s \in [0,2]$$

$$\|F_h(t)A^{-s/2}\|_{\mathcal{B}(H)} \le Ch^s, \quad t \ge 0, \quad s \in [0,2]$$

We now use $s = 2\beta$.

The trace norm

Nuclear operators on H:

$T \in \mathcal{L}_1(H)$
if there are sequences $\{a_j\}, \{b_j\} \subset H$ with $\sum_{j=1}^{\infty} \|a_j\|\|b_j\| < \infty$ and such that

$$Tx = \sum_{j=1}^{\infty} \langle x, b_j \rangle a_j, \quad \forall x \in H.$$

Banach space with norm: $\|T\|_{\mathrm{Tr}} = \inf \Big\{ \sum_{j=1}^{\infty} \|a_j\|\|b_j\| : Tx = \sum_{j=1}^{\infty} \langle x, b_j \rangle a_j \Big\}.$

For $T \in \mathcal{L}_1(H)$: $\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$ with $\{e_k\}_{k=1}^{\infty}$ ONB of H

Connection with HS: $\|T\|_{\mathrm{HS}}^2 = \mathrm{Tr}(T^*T) = \|T^*T\|_{\mathrm{Tr}}$

Linear CHC: weak convergence

Theorem

Assume:

$$\|A^{\beta-2+\alpha}Q\|_{\mathcal{B}(H)} < \infty, \quad \|A^{-\alpha}\|_{\mathrm{Tr}} < \infty,$$

for some $\beta \in (0,1]$, $\alpha > 0$ *with* $0 \le \beta - 2 + \alpha \le 1$.

Then, for $G \in C_{\mathrm{b}}^2(H,\mathbb{R})$,

$$\mathbf{E}\Big(G(X_h(T))\Big) - \mathbf{E}\Big(G(X(T))\Big) \le Ch^{2\beta}|\log(h)|.$$

Twice the rate of strong convergence: if $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 < \infty$ then

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \le Ch^{\beta}|\log(h)|\,\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}, \quad t \ge 0$$

Comparison of norms:

$$\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 = \|A^{\frac{\beta-2}{2}}QA^{\frac{\beta-2}{2}}\|_{\mathrm{Tr}} \le \|A^{\beta-2}Q\|_{\mathrm{Tr}} \le \|A^{\beta-2+\alpha}Q\|_{\mathcal{B}(H)}\|\|A^{-\alpha}\|_{\mathrm{Tr}}$$

Linear CHC: weak convergence

Generalization of Debussche and Printems (2009) (heat equation)

Kovács, Larsson, and Lindgren, preprint (2009)

(also applied to the heat and wave equations)

Hausenblas, preprint (2009) (wave equation)

Cahn-Hilliard-Cook equation

$$\begin{cases} \mathrm{d}X + A^2 X\,\mathrm{d}t = -Af(X)\,\mathrm{d}t + \mathrm{d}W, \quad t > 0 \\ X(0) = X_0 \end{cases}$$

The mild solution is given by the equation:

$$\begin{aligned} X(t) &= \mathrm{e}^{-tA^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s) \\ &= Y(t) + W_A(t) \end{aligned}$$

The stochastic convolution is now known:

$$W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s)$$

Remains to solve the random evolution problem:

$$\begin{cases} \dot{Y} + A^2 Y + Af(Y + W_A) = 0, \quad t > 0 \\ Y(0) = X_0. \end{cases}$$

Cahn-Hilliard-Cook equation

Da Prato and Debussche (1996)

- Assume common eigenbasis for A and Q.
- Galerkin approximation, energy estimates, $Y_N \to Y$.
- For $\|A^{\frac{1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ they obtain $Y \in C([0,T], \dot{H}^1)$ a.s.
- But $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ implies $W_A(t) \in L_2(\Omega, \dot{H}^\beta)$, that is, we have $\beta = 3$.

Our goal: to prove higher regularity of this solution and to prove convergence estimates.

We use the mild formulation:

$$Y(t) = \mathrm{e}^{-tA^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} A f\big(Y(s) + W_A(s)\big)\,\mathrm{d}s$$

Cahn-Hilliard-Cook equation

$$Y(t) = \mathrm{e}^{-tA^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af\big(Y(s) + W_A(s)\big)\,\mathrm{d}s$$

Controlling the non-linearity: $f(s) = s^3 - s$

- $\|Af(u)\| \le C\big(1 + \|u\|_{H^1}^2\big)\|u\|_{H^3}$
- $|f(u) - f(v)|_{-1} \le C\big(1 + \|u\|_{H^1}^2 + \|v\|_{H^1}^2\big)\|u - v\|$

Useful to bound $\|X(t)\|_{H^1}$ and $\|X_h(t)\|_{H^1}$.

Cahn-Hilliard-Cook equation

Energy functional:

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u \in H^1, \quad F(s) = \tfrac{1}{4}s^4 - \tfrac{1}{2}s^2,$$

Deterministic case: $J(X(t)) \le J(X_0), \quad t \ge 0 \quad$ (Lyapunov functional)

Stochastic case: if $\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$, then

$$\mathbf{E}\big[J(X(t))\big] \le C(t), \quad \mathbf{E}\big[J(X_h(t))\big] \le C(t), \quad t \ge 0.$$

Hence: $\mathbf{E}\big[\|X(t)\|_{H^1}^2\big] \le C(t), \quad \mathbf{E}\big[\|X_h(t)\|_{H^1}^2\big] \le C(t), \quad t \ge 0.$

Proof. Apply Itô's formula to $J(X_N(t))$ and $J(X_h(t))$.

Generalization of Da Prato and Debussche (1996):

- do not assume common eigenbasis for A and Q.
- do not assume max-norm bound for the eigenbasis of Q: $\|e_j\|_{L_\infty(\mathcal{D})} \le C$.
- the growth $C(t)$ is quadratic instead of exponential.
- same bound for X_h.

Cahn-Hilliard-Cook equation

$$\mathbf{E}\big[\|X(t)\|_{H^1}^2\big] \le C(t), \quad t \ge 0$$

$$\|Af(X)\| \le C\big(1 + \|X\|_{H^1}^2\big)\|X\|_{H^3}$$

$$\mathbf{E}\Big[\|X\|_{H^1}^2\|X\|_{H^3}\Big] \le \mathbf{E}\Big[\|X\|_{H^1}^4\Big]^{\frac{1}{2}}\mathbf{E}\Big[\|Y + W_A\|_{H^3}^2\Big]^{\frac{1}{2}} \quad \text{No good.}$$

Chebyshev's inequality:

for each $T > 0$ and $\epsilon \in (0,1)$ there are K_T and $\Omega_\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_\epsilon) \ge 1 - \epsilon$ and such that

$$\|X(t)\|_{H^1}^2 + \|X_h(t)\|_{H^1}^2 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t \in [0, T]$$

Now we can control the nonlinearity pointwise on $\Omega_\epsilon \times [0, T]$.

Deterministic linear Cahn-Hilliard: error estimate

$$\begin{cases} u(t) \in H^1(\mathcal{D}), \quad u(0) = u_0 \\ \langle \dot{u}, \chi \rangle + \langle \nabla v, \nabla \chi \rangle = 0 \quad \forall \chi \in H^1(\mathcal{D}), \ t > 0 \\ \langle v, \chi \rangle = \langle \nabla u, \nabla \chi \rangle + \langle g, \chi \rangle \quad \forall \chi \in H^1(\mathcal{D}), \ t > 0 \end{cases}$$

$$\begin{cases} u_h(t) \in S_h, \quad u_h(0) = P_h u_0 \\ \langle \dot{u}_h, \chi \rangle + \langle \nabla v_h, \nabla \chi \rangle = 0 \quad \forall \chi \in S_h, \ t > 0 \\ \langle v_h, \chi \rangle = \langle \nabla u_h, \nabla \chi \rangle + \langle g, \chi \rangle \quad \forall \chi \in S_h, \ t > 0 \end{cases}$$

Time-derivative-free error estimate:

$$\begin{aligned} \|u_h(t) - u(t)\| &\leq Ch^2 \Big(|\log(h)| \sup_{s \in [0,t]} |u(s)|_2 + \Big(\int_0^t |v(s)|_2^2 \, \mathrm{d}s \Big)^{\frac{1}{2}} \Big) \\ &\leq Ch^2 |\log(h)| \Big(|u_0|_2^2 + \int_0^t |g(s)|_2^2 \, \mathrm{d}s \Big)^{\frac{1}{2}} \end{aligned}$$

To be used with $g(t) = f(X(t))$ so that $u(t) = Y(t)$.

Cahn-Hilliard-Cook equation

Regularity and error estimate on Ω_ϵ:

$$\|X(t)\|_{H^3} \leq C(\epsilon^{-1}K_T, T) \qquad \text{on } \Omega_\epsilon,\ t \in [0, T]$$

$$\|X_h(t) - X(t)\| \leq C(\epsilon^{-1}K_T, T)h^2|\log(h)| \qquad \text{on } \Omega_\epsilon,\ t \in [0, T]$$

Strong convergence (without known rate):

$$\max_{t\in[0,T]} \mathbf{E}\big[\|X_h(t) - X(t)\|^2\big] \to 0 \quad \text{as } h \to 0.$$

Kovács, Larsson, and Mesforush, preprint, (2010)