Scalar conservation laws with stochastic forcing

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SPDEs: Approximation, Asymptotics and Computation, July 2010 Cambridge

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Introduction

Scalar conservation laws with stochastic forcing

First-order scalar conservation laws with stochastic forcing

$$\begin{cases} du + \mathrm{div}(A(u))dt = \Phi(u)dW(t), & x \in \mathbb{T}^N, t \in (0,T), \\ u(0) = u_0 \in L^\infty(\mathbb{T}^N) \end{cases}$$

Flux $A \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$, noise $\Phi(u)dW(t)$ (to be specified), the unknown

$$u\colon (x,t) \mapsto u(x,t) \in \mathbb{R}$$

is scalar.

First-order scalar conservation law

We will suppose that $a(u) := A'(u)$ has at most polynomial growth.

First Example: Transport equation $a(u) = \mathrm{cst}$,

$$\partial_t u + a \cdot \nabla u = 0.$$

Solution: transport of the profile U:

$$u(x,t) = U(x - ta).$$

First-order scalar conservation law

Second example: Inviscid Burgers' equation $a(u) := a_0 u$, $a_0 = \mathrm{cst}$,

$$\partial_t u + a_0 u \cdot \nabla u = 0.$$

Solution: *non-linear* transport of the profile U:

$$u(x,t) = U(x - tu(x,t)a_0).$$

Multiplicative and additive noise

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis,

$$\Phi(u)W = \sum_{k\geq 1} g_k(u)\beta_k,$$

where, $g_k \in C(\mathbb{T}^N \times \mathbb{R})$, with the following bounds:

$$\mathbf{G}^2(x,u) = \sum_{k\geq 1} |g_k(x,u)|^2 \leq D_0(1+|u|^2),$$

$$\sum_{k\geq 1} |g_k(x,u) - g_k(y,v)|^2 \leq D_1(|x-y|^2 + |u-v|h(|u-v|)),$$

where $x, y \in \mathbb{T}^N$, $u, v \in \mathbb{R}$, and h is a continuous non-decreasing function on $\mathbb{R}_+$ with $h(0) = 0$.

Additive noise

The problem rewrites

$$\partial_t w + \mathrm{div}(A(w+J)) = 0,$$

for

$$w := u - J, \quad J(x,t) := \int_0^t \sum_{k\geq 1} g_k(x,s)\,d\beta_k(s).$$

References:

- Kim 2003: $1D$, Cauchy Problem ($x \in \mathbb{R}$),
- Vallet, Wittbold 2009: multi D, Cauchy-Dirichlet Problem ($x \in$ Domain).

Multiplicative noise

References:

- Feng, Nualart 2008: $1D$ existence, multi-D uniqueness (of strong solutions), Cauchy Problem ($x \in \mathbb{R}^N$),
- Debussche, Vovelle 2010: multi-D, Periodic Problem ($x \in \mathbb{T}^N$).

Main result

Theorem. There exists a unique solution *(to be specified)*, which is the limit of the viscous approximation: for all $1 \leq p < +\infty$,

$$\lim_{\eta \to 0} \mathbb{E} \| u^\eta - u \|_{L^p(\mathbb{T}^N \times (0,T))} = 0,$$

where u^η is solution to

$$du^\eta + \mathrm{div}(A(u^\eta))dt - \eta \Delta u^\eta dt = \Phi_\eta(u^\eta) dW(t).$$

Solution (kinetic formulation)

Definition. A measurable $u\colon \mathbb{T}^N \times (0,T) \times \Omega \to \mathbb{R}$ is solution if $(u(t))$ is predictible,

$$\|u(t)\|_{L^p(\Omega\times\mathbb{T}^N)} \leq C_p \text{ for a.e. } t \in (0,T), \quad 1 \leq p < +\infty,$$

there exists a kinetic measure m such that $f := \mathbf{1}_{u>\xi}$ satisfies a.s.

$$df + a(\xi)\cdot\nabla f dt + g_k(x,\xi)\partial_\xi f d\beta_k(t) = \partial_\xi\left(m - \frac{1}{2}|\mathbf{G}(x,\xi)|^2\delta_{u=\xi}\right)dt,$$

$$f(0) = \mathbf{1}_{u_0>\xi},$$

in the weak sense (weak w.r.t. *time* and *space* and ξ-variables).

Kinetic formulation

Non-linear equation, kinetic formulation

Consider the equation

$$\partial_t u + \mathrm{div}(A(u)) = 0.$$

Set

$$a(u) = A'(u),$$

to rewrite (1^{st}-order PDE, non-conservative version)

$$\partial_t u + a(u) \cdot \nabla u = 0,$$

with solution

$$u(x,t) = U(x - ta(u(x,t))).$$

Kinetic formulation

Set

$$f(x,t,\xi) = \mathbf{1}_{u(x,t)>\xi}, \quad \xi \in \mathbb{R},$$

(characteristic function of the subgraph of u).

Aim: derive the equation satisfied by f.

Kinetic formulation

If u is smooth and $\eta \in C^2(\mathbb{R})$, then, by the chain-rule,

$$\partial_t \eta(u) + \mathrm{div}\Phi(u) = 0, \quad \Phi'(u) = \eta'(u)a(u).$$

In terms of f, assuming $\eta(-\infty) = 0$,

$$\eta(u) = \langle f, \eta' \rangle := \int_{\mathbb{R}} f(\xi)\eta'(\xi)d\xi, \quad \Phi(u) = \langle a(\xi)f, \eta' \rangle,$$

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hence

$$(\partial_t + a(\xi)\cdot\nabla)f = 0, \quad x \in \mathbb{T}^N, t \in (0,T), \xi \in \mathbb{R}.$$

Picture: shocks

Shock, entropy formulation

Theorem(Lions, Perthame, Tadmor 94) Let $u_0 \in L^\infty(\mathbb{T}^N)$. There exists a unique kinetic solution to the Cauchy Problem

$$\begin{cases} \partial_t u + \operatorname{div}(A(u)) = 0, & x \in \mathbb{T}^N, t \in (0,T), \\ u(0,x) = u_0(x), & x \in \mathbb{T}^N, \end{cases}$$

in the sense that

$$u \in L^\infty(\mathbb{T}^N \times (0,T)) \cap C([0,T]; L^1(\mathbb{T}^N)), \quad u(0) = u_0,$$

and: there exists a non-negative measure[3] m on $\mathbb{T}^N \times [0,T] \times \mathbb{R}$ such that

$$(\partial_t + a(\xi) \cdot \nabla) f = \partial_\xi m.$$

[3]supported in $\mathbb{T}^N \times [0,T] \times [-R,R]$

Kinetic formulation for the parabolic approximation

The kinetic formulation of the parabolic equation

$$du^\eta + \mathrm{div}(A(u^\eta))dt - \eta\Delta u^\eta dt = 0$$

is

$$(\partial_t + a(\xi)\cdot\nabla - \eta\Delta)f^\eta = \partial_\xi m^\eta,$$

with

$$f^\eta = \mathbf{1}_{u^\eta>\xi}, \quad m^\eta = \eta|\nabla u^\eta|^2\delta_{u^\eta=\xi}.$$

Note: m^η *bounded in mass* by the standard *energy estimate* for the parabolic problem.

L^1-contraction

Kinetic formulation: the evolution of u by

$$\partial_t u + \mathrm{div}(A(u)) = 0 \tag{1}$$

is

$$\underbrace{(\partial_t + a(\xi) \cdot \nabla) f}_{\text{Transport}} = \underbrace{\partial_\xi m, \quad f = \mathbf{1}_{u>\xi}}_{\text{constraint}}.$$

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Theorem The difference $\|u_1 - u_2\|_{L^1(\mathbb{T}^N)}(t)$ of two kinetic solutions u_1, u_2 to (1) is non-increasing with t.

L^1-contraction

Proof (Perthame, 98) By

$$(u_1 - u_2)^+ = \int_{\mathbb{R}} f_1 \bar{f}_2 d\xi, \quad f_1 = \mathbf{1}_{u_1 > \xi}, \quad \bar{f}_2 = \mathbf{1}_{u_2 \leq \xi} = 1 - f_2,$$

and

$$(\partial_t + a(\xi) \cdot \nabla) f_1 = \partial_\xi m_1, \quad (\partial_t + a(\xi) \cdot \nabla) \bar{f}_2 = -\partial_\xi m_2,$$

we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}^N} (u_1 - u_2)^+ dx = & \int_{\mathbb{T}^N} \int_{\mathbb{R}} -\partial_\xi \bar{f}_2 m_1 + \partial_\xi f_1 m_2 \\ & \int_{\mathbb{T}^N} \int_{\mathbb{R}} -\delta_{u_2 = \xi} m_1 - \delta_{u_1 = \xi} m_2 \leq 0. \end{aligned}$$

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2.f. L^1-contraction property

Equation with source term

The kinetic formulation for

$$\partial_t u + \mathrm{div}(A(u)) = S(x,t,u)$$

is

$$(\partial_t + a(\xi)\cdot\nabla + S(x,t,\xi)\partial_\xi)f = \partial_\xi m.$$

Kinetic formulation of ODEs

For solutions to $\dot{u} = S(t, u)$, the chain rule gives

$$\frac{d}{dt}\eta(u) = \eta'(u)S(t, u).$$

In terms of kinetic function:

$$\frac{d}{dt}\int_{\mathbb{R}} f(\xi)\eta'(\xi)d\xi = \int_{\mathbb{R}} \delta_{u=\xi} S(t, \xi)\eta'(\xi)d\xi,$$

i.e., using $\delta_{u=\xi} = -\partial_\xi f$,

$$\frac{d}{dt}\langle f, \eta' \rangle = -\langle S\partial_\xi f, \eta' \rangle,$$

and

$$\partial_t f + S(t, \xi)\partial_\xi f = 0.$$

Kinetic formulation of SDEs

Let β be a brownian motion over $\mathbb{R}$. For solutions to $du = S(t,u)d\beta(t)$, Itô Formula gives

$$d\eta(u) = \eta'(u)S(t,u)d\beta(t) + \frac{1}{2}|S(t,u)|^2\eta''(u)dt.$$

In terms of kinetic function:

$$d\langle f, \eta'\rangle = \langle \delta_{u=\xi} S(t,\xi)d\beta(t), \eta'\rangle + \frac{1}{2}\langle \delta_{u=\xi}|S(t,\xi)|^2, \partial_\xi \eta'\rangle,$$

i.e., using $\delta_{u=\xi} = -\partial_\xi f$,

$$df + S(t,\xi)\partial_\xi f d\beta(t) = \frac{1}{2}\partial_\xi(|S(t,\xi)|^2\partial_\xi f)dt.$$

Solution (kinetic formulation)

Definition. A measurable $u\colon \mathbb{T}^N \times (0,T) \times \Omega \to \mathbb{R}$ is solution if $(u(t))$ is predictible,

$$\|u(t)\|_{L^p(\Omega\times\mathbb{T}^N)} \leq C_p \text{ for a.e. } t \in (0,T), \quad 1 \leq p < +\infty,$$

there exists a kinetic measure m such that $f := \mathbf{1}_{u>\xi}$ satisfies a.s.

$$df + a(\xi)\cdot\nabla f dt + g_k(x,\xi)\partial_\xi f d\beta_k(t) = \partial_\xi\left(m - \frac{1}{2}|\mathbf{G}(x,\xi)|^2\delta_{u=\xi}\right)dt,$$
$$f(0) = \mathbf{1}_{u_0>\xi},$$

in the weak sense (weak w.r.t. *time* and *space* and ξ-variables).

Kinetic measure

Definition. We say that a map m from Ω to $\mathcal{M}_b^+(\mathbb{T}^N \times [0,T] \times \mathbb{R})$ (non-negative bounded measures) is a kinetic measure if

1. m is measurable, in the sense that for each $\phi \in C_b(\mathbb{T}^N \times [0,T] \times \mathbb{R})$, $\langle m, \phi \rangle : \Omega \to \mathbb{R}$ is,
2. m vanishes for large ξ: if $B_R^c = \{\xi \in \mathbb{R}, |\xi| \geq R\}$, then
$$\lim_{R \to +\infty} \mathbb{E} m(\mathbb{T}^N \times [0,T] \times B_R^c) = 0,$$
3. for all $\phi \in C_b(\mathbb{T}^N \times \mathbb{R})$, the process
$$t \mapsto \int_{\mathbb{T}^N \times [0,t] \times \mathbb{R}} \phi(x,\xi) dm(x,s,\xi)$$
is predictible.

Resolution

Theorem. There exists a unique solution, which is the limit of the viscous approximation: for all $1 \leq p < +\infty$,

$$\lim_{\eta \to 0} \mathbb{E}\|u^\eta - u\|_{L^p(\mathbb{T}^N \times (0,T))} = 0,$$

where u^η is solution to

$$du^\eta + \mathrm{div}(A(u^\eta))dt - \eta \Delta u^\eta dt = \Phi_\eta(u^\eta)dW(t).$$

Invariant measure

Additive noise, mass conservation

- Weinan E, Khanin, Mazel, Sinai 2000: invariant measure for $1D$, periodic, stochastically forced inviscid Burgers equation (corresponding to $A(u) = \frac{u^2}{2}$ with additive noise here).
- Dirr, Souganidis 2005: invariant measure for multi-D, periodic HJ equations with additive noise:
  $du + H(\nabla u, x)dt = dW(x, t)$.

Kinetic formulation of the parabolic approximation

The characteristic function $f^\eta := \mathbf{1}_{u^\eta > \xi}$ satisfies the estimates and equations

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu^\eta_{x,t}(\xi) dx \leq C_p, \quad t \in (0,T), \quad 1 \leq p < +\infty,$$

where $\nu^\eta_{x,t} := -\partial_\xi f^\eta(x,t,\cdot) = \delta_{u^\eta(x,t)=\xi}$, and

$$df^\eta + a(\xi) \cdot \nabla f^\eta dt - \eta \Delta f^\eta dt - g_k(x,\xi) \nu^\eta_{x,t} d\beta_k(t) = \partial_\xi \left(m^\eta - \frac{1}{2} |\mathbf{G}(x,\xi)|^2 \nu^\eta_{x,t} \right) dt,$$

in the weak sense, with $m^\eta := \eta |\nabla u^\eta|^2 \delta_{u^\eta = \xi}$.

Further estimates, convergence

Estimate in mass on m^η:

$$\|m^\eta\|_{L^2(\Omega;\mathcal{M}(\mathbb{T}^N\times[0,T]\times\mathbb{R}))} \leq C.$$

Convergence:

- $m^\eta \rightharpoonup m$,
- $\nu^\eta_{x,t} \rightharpoonup \nu_{x,t}$,
- $f^\eta(x,t,\xi) = \nu^\eta_{x,t}(\xi,+\infty) \rightharpoonup f(x,t,\xi) := \nu_{x,t}(\xi,+\infty)$.

Limit $\eta \to 0$

Passing to the limit in the linear equation

$$df^\eta + a(\xi)\cdot\nabla f^\eta dt - \eta\Delta f^\eta dt - g_k(x,\xi)\nu^\eta_{x,t} d\beta_k(t) = \partial_\xi\left(m^\eta - \frac{1}{2}|\mathbf{G}(x,\xi)|^2\nu^\eta_{x,t}\right)dt,$$

gives

$$df + a(\xi)\cdot\nabla f dt - g_k(x,\xi)\nu_{x,t} d\beta_k(t) = \partial_\xi\left(m - \frac{1}{2}|\mathbf{G}(x,\xi)|^2\nu_{x,t}\right)dt.$$

Generalized solutions, Reduction

Generalized solution

Definition A measurable function $f\colon \Omega\times\mathbb{T}^N\times[0,T]\times\mathbb{R}\longrightarrow[0,1]$ is said to be a generalized solution with initial datum if $(f(t))$ is predictible and if there exists a Young measure $\nu_{x,t}$ such that $f(x,t,\xi):=\nu_{x,t}(\xi,+\infty)$,

$$\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\xi|^p d\nu_{x,t}(\xi)dx\leq C_p,\quad t\in(0,T),\quad 1\leq p<+\infty$$

and if there exists a kinetic measure m s.t.

$$df+a(\xi)\cdot\nabla f dt-g_k(x,\xi)\nu_{x,t}(\xi)d\beta_k(t) \\ =\partial_\xi\left(m-\frac{1}{2}|\mathbf{G}(x,\xi)|^2\nu_{x,t}(\xi)\right)dt.$$

Equilibrium, strong convergence

Convergence of ν^η (where $Z = \Omega \times \mathbb{T}^N \times (0, T)$ with measure $d\lambda = d\mathbb{P} \otimes dx \otimes dt$): $\forall h \in L^1(Z), \forall \phi \in C_b(\mathbb{R})$,

$$\int_Z h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z^\eta(\xi) d\lambda(z) \to \int_Z h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z(\xi) d\lambda(z).$$

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Convergence of ν^η (where $Z=\Omega\times\mathbb{T}^N\times(0,T)$ with measure $d\lambda=d\mathbb{P}\otimes dx\otimes dt$): $\forall h\in L^1(Z),\forall\phi\in C_b(\mathbb{R})$,

$$\int_Z h(z)\int_{\mathbb{R}}\phi(\xi)d\nu^\eta_z(\xi)d\lambda(z)\to\int_Z h(z)\int_{\mathbb{R}}\phi(\xi)d\nu_z(\xi)d\lambda(z).$$

Proposition: There is equivalence beween:

- Strong convergence: (u^η) converges strongly in $L^1(Z)$,
- Structure: there exists $u\in L^1(Z)$ such that $\nu_z(\xi)=\delta_{u(z)=\xi}$ for λ-a.e. z.

Reduction, Uniqueness

Theorem (Reduction, uniqueness)

- Let f be a generalized solution starting from equilibrium: $f(0) = \mathbf{1}_{u_0 > \xi}$. Then there exists a solution u such that $f = \mathbf{1}_{u > \xi}$.
- The difference $\mathbb{E}\|u_1 - u_2\|_{L^1(\mathbb{T}^N)}(t)$ of two kinetic solutions u_1, u_2 is non-increasing with t.

Thank you !