On the stochastic non linear Schrödinger equation

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Outline

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M compact Riemanian manifold of dimension d without boundary
$Z : [0, T] \times M \to \mathbb{C}$ solution to

$$i\partial_t Z(t) + \Delta Z(t)dt = f(Z(t))dt, \quad Z(0) = z_0 \in H^1 := H^{1,2}(M)$$

Burq, Gérard and Tzvetkov (2004) proved the following:

- **Dimension** $d = 2$

$f(z) = F'(|z|^2)z$ for some polynomial F with real coefficients s.t.
$F(r) \to +\infty$ as $r \to +\infty$ (or $F(r) = ar^\delta$ with $a > 0$ or $a < 0$ and $\delta < 2$)
then the NLS equation has a unique solution in $C([0, T], H^1)$
If $Z_0 \in H^s := H^{s,2}$ with $s > 1$, then $Z_. \in C([0, T], H^s)$

- **Dimension** $d = 3$ the same result holds for $F'(r) = r$ i.e.

$$i\partial_t Z(t) + \Delta Z(t)dt = |Z(t)|^2 Z(t)dt$$

Δ Laplace Beltrami operator
$H^{s,q} = Dom((-\Delta_q)^{\frac{s}{2}})$ for $2 \le q < \infty$ and $s > 0$
shorthand notation $H^{s,2} = H^s$
$(S(t), t \in \mathbb{R})$ group $S(t) = \exp(it\Delta)$ acting on $L^2(M)$ (unitary group)
Strichartz inequalities Let p, q satisfy the scaling admissible condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$$

Let $s \ge 0$; for $Z \in H^{s+\frac{1}{p}}$ one has

$$\|S(.)Z\|_{L^p(0,T;H^{s,q})} \le C(T)\|Z\|_{H^{s+\frac{1}{p}}}$$

and for $\phi \in L^1(0, T, H^{s+\frac{1}{p}})$ if $S * \phi(t) = \int_0^t S(t-r)\phi(r)dr$

$$\Big(\int_0^T \|S * \phi(t)\|^p_{H^{s,q}} dt\Big)^{\frac{1}{p}} \le C(T) \int_0^T |\phi(t)|_{H^{s+\frac{1}{p}}} dt$$

Same constraints on admissible pairs: $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$

• **In** $\mathbb{R}^d$ the Strichartz inequality is "better"
no gain/loss of regularity, and improvement of integrability

$$\|S(.)Z\|_{L^p(0,T;L_q)} \leq C\|Z\|_{L^2}$$

known to be sharp

• **Compact** manifold: loss of regularity unavoidable

$$\|S(.)Z\|_{L^p(0,T;H^{s,q})} \leq C(T)\|Z\|_{H^{s+\frac{1}{p}}}$$

Is the "regularity loss" of $1/p$ sharp? No clear answer in general
On the sphere of dimension $d \geq 2$, $\|S(.)Z\|_{L^4([0,T]\times M)} \leq C(T)\|Z\|_{H^s}$
with $s > s(d)$ and $s(2) = \frac{1}{8}$, $s(d) = \frac{d}{4} - \frac{1}{2}$ is "sharp"

• **Fixed point argument** Let F be of degree δ, $s > \frac{d}{2} - \frac{1}{\max(\delta-1,2)}$ and $|Z_0|_{H^s} \le R$
For T small enough, equation

$$i\partial_t Z(t) + \Delta Z(t)dt = f(Z(t))dt, \quad Z(0) = z_0 \in H^1 := H^{1,2}(M)$$

has a unique solution in $C([0,T];H^s) \cap L^p(0,T;L^\infty)$
• **Maximal solution** There exists T^* such that the solution to NLS exists in $C([0,T^*);H^s) \cap L^p(0,T^*;L^\infty)$ and as $t \to T*$ with $t < T^*$, norm converges to ∞
• **Global solutions - Conservation laws** $f(z) = F'(|z|^2)z$ and F real
There exists $\tilde{C} > 0$ such that for every $t \in [0,T^*)$

$$\int_M |Z(t,x)|^2 dx = \int_M |Z_0(x)|^2 dx$$

$$\int_M |\nabla Z(t,x)|^2 dx + \int_M F(|Z(t,x)|^2) dx = \tilde{C}$$

Formal proofs; $\langle Z_1, Z_2\rangle = Z_1\overline{Z_2}$; F real-valued. **First conservation law**

$$d_t \int_M |Z(t,x)|^2 dx = 2\int_M \mathrm{Re}\langle Z(t,x), d_t Z(t,x)\rangle dx$$

$$= 2\int_M \mathrm{Re}\Big(i\langle Z(t,x), \Delta Z(t,x) - F'(|Z(t,x)|^2)Z(t,x)\rangle\Big) dx = 0$$

Second conservation law

$$d_t \int_M |\nabla Z(t,x)|^2 dx + d_t \int_M F(|Z(t,x)|^2) dx =$$

$$2\int_M \Big[\mathrm{Re}\langle \nabla Z(t,x), d_t \nabla Z(t,x)\rangle + F'(|Z(t,x)|^2)\mathrm{Re}\langle Z(t,x), d_t Z(t,x)\rangle\Big] dx$$

$$= 2\int_M \mathrm{Re}\langle -\Delta Z(t,x) + F'(|Z(t,x)|^2 Z(t,x), d_t Z(t,x)\rangle dx$$

$$= -2\int_M \mathrm{Re}\Big(i\langle d_t Z(t,x), d_t Z(t,x)\rangle\Big) dx = 0$$

$d = 2$ prove that $T^* = \infty$ in both cases
Case 1 $F(r) = ar^\delta + \sum_{\gamma<\delta} b_\gamma r^\gamma$ with $a > 0$

$$|\nabla Z(t)|_2^2 + a\|Z(t)\|_{2\delta}^{2\delta} \leq \tilde{C} + \sum_{0\leq\gamma<\delta} |b_\gamma| \|Z(t)\|_{2\gamma}^{2\gamma}$$

For $1 \leq \gamma \leq \delta - 1$; Hölder's inequality yields

$$\|Z(t)\|_{2\gamma}^{2\gamma} \leq \epsilon \|Z(t)\|_{2\delta}^{2\delta} + C|Z_0|_2^2/\epsilon$$

Case 2 $F(r) = ar^\delta$ with $a < 0$, apply the Gagliardo Nirenberg inequality

$$\|Z(t)\|_{2\delta}^{2\delta} \leq C|\nabla Z(t)|_2^{(\delta-1)d} |Z(t)|_2^{2\delta-(\delta-1)d} \leq \epsilon |\nabla Z(t)|_2^2 + (C/\epsilon)|Z_0|_2^{n(\delta)}$$

if $\delta < 2$

• Add a random perturbation driven by a "real-valued Wiener process W" with diffusion coefficient $g(Z_s)$
used in equations modelling the propagation of nonlinear dispersive waves in nonhomogeneous random media, in optics
Try to prove the existence and uniqueness of a solution
$Z \in C([0,T], H^1)$
• Solved by de Bouard and Debussche on $\mathbb{R}^d$ and in the particular case $f(z) = \lambda |z|^{2\sigma}\, z$ (i.e., $F(r) = r^{\sigma+1}$) and $g(z) = z$
In their setting, W is $L^2(\mathbb{R}^d)$-valued

$$W(t,w,\omega) = \sum_k \phi e_k(x) \beta_k(t,\omega)$$

where $\beta_k, k \geq 0$ are independent real-valued Brownians, (e_k) is an ONB of $L^2(\mathbb{R}^d)$ and ϕ is an operator such that $\phi\phi^*$ is trace class.

de Bouard Debussche used **Stratonovich integral** $Z(t) \circ dW_t$

- prove existence of a maximal solution by some fixed point argument
- Stratonovich integrals allow to prove the first conservation law

$|Z(t)|_2 = Z_0$ a.s.

prove existence of a global solution $Z \in C([0,T]; H^1)$ for
$f(z) = \lambda |z|^{2\sigma} z$ if either $\lambda > 0$ or $\sigma < 2/d$ and other assumptions .
a Hamiltonian similar to that of the second conservation law reduces to the stochastic term

requires more assumptions on the noise which is a true Wiener process on $H^{1,\alpha} \cap H^1$ for $\alpha > 2d$

Given $Z(0) = z_0 \in H^1$; find a maximal solution to

$$i\partial_t Z(t) + \Delta Z(t)dt = f(Z(t))dt + g(Z(t))dW(t)$$

f is polynomial of degree β (previous case $\beta = 2\delta - 1$)

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2},\ p > \max(\beta - 1, 2),\ s > \frac{d}{2} - \frac{1}{p},\ \hat{s} = s - \frac{1}{p} > \frac{d}{q}$$

W cylindrical Brownian motion on Hilbert space K
Let $G(Z) = g \circ Z$; suppose $G : H^s \cap H^{\hat{s},q} \to L_{(2)}(K, H^s)$
where $L_{(2)}(K, H^s)$ denotes the set of **Hilbert-Schmidt** operators

Proposition

Suppose that $G : H^s \cap H^{\hat{s},q} \to L_{(2)}(K, H^s)$ *is such that for* $Z, \zeta \in H^s \cap H^{\hat{s},q}$:

(i) $\|G(Z)\|_{L_{(2)}(K,H^s)} \leq C(1 + |Z|_{H^s})\,(1 + \|Z\|_{H^{\hat{s},q}})$

(ii)

$$\begin{aligned}\|G(Z) - G(\zeta)\|_{L_{(2)}(K,H^s)} \leq & C|Z-\zeta|_{H^s}\left[1 + \|Z\|_{H^{\hat{s},q}} + \|\zeta\|_{H^{\hat{s},q}}\right] \\ & + C\|Z-\zeta\|_{H^{\hat{s},q}}\left[1 + |Z|_{H^s} + |\zeta|_{H^s}\right]\end{aligned}$$

Then if $s > \frac{d}{2} - \frac{1}{p}$, **for "small"** T *there exists a unique (mild) solution in* $C([0,T]; H^s) \cap L^p(0,T; H^{\hat{s},q})$ *to the SLNS equation* $dZ(t) = i\Delta Z(t)dt - if(Z(t))dt - ig(Z(t))dW(t)$, $Z(0) \in H^s$

Existence of a local solution on compact manifolds
The fixed point argument

For $n \geq 1$, set $\theta_n(.) = \theta(\frac{\cdot}{n})$ where $\theta : [0, \infty) \to [0,1]$ be of class C^∞, compactly supported non increasing function

$$\inf_{x\in\mathbb{R}} \theta'(x) \geq -1, \quad \theta(x) = 1 \text{ iff } x \in [0,1] \quad \text{and } \theta(x) = 0 \text{ iff } x \in [2,\infty).$$

Let $Z_0 \in H^s$, $S(t) = \exp(it\Delta)$ be the Schrödinger semigroup and rewrite the mild form of the solution

$$Z(t) = S(t)(Z_0) + \int_0^t S(t-\tau) f(Z(\tau)) d\tau + \int_0^t S(t-\tau) g(Z(\tau)) dW_\tau$$

truncating the coefficients as follows: find $Z^n \in Y_T$ a.s. such that

$$\begin{aligned} Z^n(t) =& S(t)(Z_0) + \int_0^t S(t-\tau) \left[\theta_n(|Z^n|_{Y_\tau}) f(Z^n(\tau))\right] d\tau \\ &+ \int_0^t S(t-\tau)\left[\theta_n(|Z^n|_{Y_\tau}) g(Z^n(\tau))\right] dW(\tau) \end{aligned}$$

Fix $n > 0$ and introduce the operator $\Phi_{T,n}$ on Y_T where

$$Y_t = C([0,t]; H^s) \cap L^p(0,t; H^{\hat{s},q}), \quad \hat{s} = s - \frac{1}{p} > \frac{d}{q}$$

$$\|u\|_{Y_t} = \sup_{0 \le \tau \le t} |u(\tau)|_{H^s} + \Big(\int_0^t |u(\tau)|^p_{H^{\hat{s},q}} d\tau \Big)^{\frac{1}{p}}, \ \forall t \in [0,T]$$

$$\Phi_{T,n}(Z)(t) = \int_0^t S(t-\tau)\big[\theta_n(|Z^n|_{Y_\tau}) f(Z^n(\tau))\big]\, d\tau$$

Lemma

There exists constants $C(T)$ (non decreasing), C_n and $C_{n,T}$ such that if $1/p + 1/p^* = 1$

$$\|\Phi_{T,n}(u)\|_{Y_T} \le C_{n,T}\Big(1 + \sup_{\tau\in[0,T]} |u(\tau)|^{p^*}_{H^s} + \Big(\int_0^T |u(\tau)|^p_{L^\infty} d\tau\Big)^{\beta-1}\Big),$$

$$\|\Phi_{T,n}(u_2) - \Phi_{T,n}(u_1)\|_{Y_T} \le C_n C(T) T^\alpha \|u_1 - u_2\|_{Y_T} \quad \text{with } \alpha > 0$$

The above inequalities come from ($\gamma = \frac{p-(\beta-1)}{p}$)

$$\int_0^T |f(Z(\tau))|_{H^s} d\tau \le CT + CT^{\frac{\gamma}{p}} \sup_\tau |Z(\tau)|_{H^s} \Big(\int_0^T |u(\tau)|_\infty^p d\tau \Big)^{\frac{\beta-1}{p}}$$

and

$$\int_0^T \Big| \theta_n(|u_2|_{Y_\tau}) f(u_2(\tau)) - \theta_n(|u_1|_{Y_\tau}) f(u_1(\tau)) \Big|_{H^s} d\tau$$
$$\le C\big[T + nT^{p^*} + n^{\beta-1} T^\gamma \big] \|u_1 - u_2\|_{Y_T}$$

the fact that $S(t) = e^{it\Delta}$ is unitary on H^s
the Strichartz inequality ($\hat{s} = s - \frac{1}{p}$)
$\|S(.)Z\|_{L^p(0,T;H^{\hat{s},q})} \le C(T)\|Z\|_{H^s}$ which implies

$$\Big(\int_0^T \|S * \phi(t)\|_{H^{\hat{s},q}}^p dt \Big)^{\frac{1}{p}} \le C(T) \int_0^T |\phi(t)|_{H^s} dt$$

Fix $n > 0$ and introduce $\Psi_{T,n}$ on

$$L^p(Y_T) := \Big\{ \text{prog. meas. } u \ : \ \mathbb{E}\Big(\sup_{0\le t\le T} |u(t)|^p_{H^s} + \int_0^T \|u(t)\|^p_{H^{\hat{s},q}} dt \Big) \Big\}$$

defined by $\Psi_{T,n}(Z)(t) = \int_0^t S(t-\tau)\big[\theta_n(|Z^n|_{Y_\tau}) g(Z^n(\tau))\big]\, dW(\tau)$.
Set $\eta(t) = \theta_n(|Z^n|_{Y_t}) g(Z^n(t))$

Lemma

For every progressively measurable process η with $(\hat{s} = s - \frac{1}{p})$ $\mathbb{E}\int_0^T \|\eta(t)\|^p_{L_{(2)}(K,H^s)} dt < \infty$ and $J\eta(t) = \int_0^t S(t-\tau)\eta(s)dW(s)$, then

$$\mathbb{E}\int_0^T \|J\eta(t)\|^p_{H^{\hat{s},q}}\, dt \le C_T \mathbb{E}\int_0^T \|\eta(t)\|^p_{L_{(2)}(K,H^s)}\, dt$$

Definition

Let K Hilbert space, $\mathcal{Y}$ martingale type 2 Banach space; a linear operator $L : K \to \mathcal{Y}$ is Radonifying if for any ONB (e_k) of K and any sequence (β_k) of iid N(0,1) random variables, the series $\sum_{k\geq 1} \beta_k L e_k$ converges in $L^2_{\tilde{\mathbb{P}}}(\tilde{\Omega};\mathcal{Y})$ (or $\tilde{\mathbb{P}}$ a.s.) and $\|L\|^2_{R(K,\mathcal{Y})} := \tilde{\mathbb{E}}\|\sum_k \beta_k K e_k\|^2_{\mathcal{Y}}$

Then if $\mathcal{Y} = H^{\hat{s},q}$ (or $W^{\hat{s},q}$) for $q \in [2,\infty)$ and if (X_t) is predictable with $X \in L^2(0,T;R(K,\mathcal{Y}))$, the stochastic integral $\int_0^t X_s dW(s)$ can be defined as an element of $L^2(0,T;\mathcal{Y})$ (extended from step processes to $L^2(0,T;R(K,\mathcal{Y}))$

Results from Dettweiler, Neidhardt, Brzezniak, Peszat, Ondrejat

Burkholder-Davies-Gundy inequality

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} \Big\| \int_0^s X_r dW(r) \Big\|^p_{\mathcal{Y}} \Big) \leq C_p(\mathcal{Y}) \mathbb{E}\Big(\int_0^t \|X_s\|^2_{R(K,\mathcal{Y})} ds \Big)^{\frac{p}{2}}$$

$$\mathbb{E}\int_0^T \|J\xi(t)\|_{\mathcal{Y}}^p\,dt \le C_p\int_0^T \mathbb{E}\Big(\int_0^t \|S(t-s)\xi(s)\|^2_{\mathrm{R}(K,\mathcal{Y})}\,ds\Big)^{p/2}dt$$

$$\le C_p T^{p/2-1}\int_0^T \mathbb{E}\Big(\int_s^T \|S(t-s)\xi(s)\|^p_{\mathrm{R}(K,\mathcal{Y})}\,dt\Big)ds$$

$$\le C_p T^{p/2-1}\int_0^T \mathbb{E}\Big(\int_0^T \|S(r)\xi(s)\|^p_{\mathrm{R}(K,\mathcal{Y})}\,dr\Big)ds$$

For $\Lambda \in R(K,H^s) = L_{(2)}(K,H^s)$, using the Kahane-Khintchine ineq. (e_j OBN of K, β_j iid $N(0,1)$ on $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$) and the Strichartz ineq.

$$\int_0^T \|S(r)\Lambda\|^p_{R(K,\mathcal{Y})}dr \le C\int_0^T \tilde{\mathbb{E}}\Big\|\sum_j \beta_j S(r)\Lambda e_j\Big\|^p_{\mathcal{Y}}dr$$

$$\le C\tilde{\mathbb{E}}\Big\|S(r)\sum_j \beta_j\Lambda e_j\Big\|^p_{L^p(0,T;W^{\hat{s},q})} \le C\tilde{\mathbb{E}}\Big\|\sum_j \beta_j \Lambda e_j\Big\|^p_{H^s} \le C\|\Lambda\|^p_{R(K,H^s)}$$

- Recall that

$$Y_t = C([0,t]; H^s) \cap L^p(0,t; W^{\hat{s},q})$$

The stochastic Strichartz estimate ($H^{\hat{s},q} \subset W^{\hat{s},q}$) proves that $\Psi_{T,n}$ is Lipschitz on $L^p(Y_T)$ (with constant depending on n and T). For fixed n, the Lipschitz constant goes to 0 as $T \to 0$

- Fix $n > 0$; for T small enough, the map $\Phi_{T,n} + \Psi_{T,n}$ is a **contraction of** $L^p(Y_T)$ Thus there exists a unique solution Z^n to the NLS with truncated coefficients in $L^p(Y_T)$

- There exists a stopping time $T_n^* \leq T$ such that for $\tau \in [0, T_n^*)$, the solution Z^n is unique in $C([0,\tau], H^s)$ with some control of the $L^p(0,\tau; W^{\hat{s},q})$ norm
The sequence T_n^* is non decreasing and due to uniqueness, $Z^n(t) = Z^{n+1}(t)$ on $[0, T_n^*)$.

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  For fixed n, the Lipschitz constant goes to 0 as $T \to 0$

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The stochastic Strichartz estimate ($H^{\hat{s},q} \subset W^{\hat{s},q}$) proves that $\Psi_{T,n}$ is Lipschitz on $L^p(Y_T)$ (with constant depending on n and T).
For fixed n, the Lipschitz constant goes to 0 as $T \to 0$
- Fix $n > 0$; for T small enough, the map $\Phi_{T,n} + \Psi_{T,n}$ is a **contraction of** $L^p(Y_T)$ Thus there exists a unique solution Z^n to the NLS with truncated coefficients in $L^p(Y_T)$
- There exists a stopping time $T_n^* \leq T$ such that for $\tau \in [0, T_n^*)$, the solution Z^n is unique in $C([0,\tau], H^s)$ with some control of the $L^p(0,\tau; W^{\hat{s},q})$ norm
The sequence T_n^* is non decreasing and due to uniqueness, $Z^n(t) = Z^{n+1}(t)$ on $[0, T_n^*)$.

Theorem

Suppose that $d = 2$, $Z_0 \in H^1$
(i) $f(z) = F'(|z|^2)z$ *such that*
* *either* $F(r) = ar^\delta$ *with* $a > 0$*, or* $a < 0$ *and* $\delta < 2$
* *or* F *polynomial of degree* δ *with* $F(r) \to +\infty$ *as* $r \to +\infty$
(ii) $g(z) = \tilde{g}(|z|^2)z$ *such that* $\tilde{g}$ *is of class* C^3
with sublinear growth condition and decay of derivatives of order 1 to 3
(e.g. $\tilde{g}(r) = C_1 + C_2\sqrt{r+1}$*)*
(iii) the noise $W(t)$ *"takes values in* $H^{1,\alpha}$*" for* $\alpha > 2$
Then for every $T > 0$ *the NLS equation:* $Z(0) = Z_0$,

$$idZ(t) + \Delta Z(t)dt = F'(|Z(t)|^2)Z(t)dt + \tilde{g}(|Z(t)|^2)Z(t) \circ dW_t$$

has a unique solution in $C([0,T], H^1)$

Let $Z_0 \in H^1$ and set $s = 1$. Prove that $T^* = \lim_n T_n^* = T$ a.s.
* **Specific form of the diffusion and drift coefficients**:
$f(z) = F'(|z|^2)z$ and $g(z) = \tilde{g}(|z|^2)z$
* **Stratonovich** formulation of the stochastic integral

$$i\partial_t Z_t + \Delta Z_t = f(Z_t)dt + g(Z_t) \circ dW_t$$

Requires **more assumptions on** W **and on** g
K Hilbert space, $L : K \to H^{1,\alpha}$ Radonifying with $\alpha > 2$ (means that W takes values in $H^{1,\alpha}$ and underlying Gaussian measure γ on $H^{1,\alpha}$)
Choose p, q admissible pair $2 < \alpha < q < 2(\alpha - 1)$; use "bounds" on $\tilde{g}$
$\mathcal{G} : H^1 \cap W^{1-\frac{1}{p},q} \to L(H^{1,\alpha}, H^1 \cap W^{1-\frac{1}{p},q})$ defined by
$\mathcal{G}(Z)(\phi) = -i\tilde{g}(|Z|^2)Z\phi$ is of class C^1

then $\mathcal{G}'(Z)\mathcal{G}(Z) \in L_2(H^{1,\alpha}, H^1 \cap W^{1-\frac{1}{p},q})$ is such that
$\mathcal{G}'(Z)\mathcal{G}(Z)(\phi,\psi) = -\tilde{g}(|Z|^2)^2 Z\phi\psi$
for $Z(.) \in L^2(0,T; H^1 \cap W^{1-\frac{1}{p},q})$ prog. meas.

$$\mathcal{G}(Z(t)) \circ dW_t = \mathcal{G}(Z(t))dW_t + \frac{1}{2} Trace_K\big(\mathcal{G}'(Z(t))\mathcal{G}(Z(t))\big)dt$$

$$Trace_K(\Lambda) = \int_{H^{1,\alpha}} \Lambda(x,x)\gamma(dx) = \sum_k \Lambda(L(e_k), L(e_k))$$

$$\partial_t Z_t = i\,\Delta Z_t - i\,F'(|Z_t|^2)Z_t dt - i\,\tilde{g}(|Z_t|^2)Z_t dW_t - \frac{1}{2}\tilde{g}(|Z_t|^2)^2 Z_t dt$$

Step 1: Prove the first conservation law $|Z(t\wedge\tau)|_{L^2(M)} \leq |Z(0)|_{L^2(M)}$ a.s. for $\tau < T^*$
Use the **Itô formula** for $|Z(t\wedge\tau)|_{L^2(M)}$ where $Z(t)$ is a **mild solution**
Use **Yosida approximation** fix $R > 0$ and set $(Z_t^R \in Dom(\Delta))$

$$\partial_t Z_t^R = i(Id + i\Delta/R)^{-1}\Delta Z_t^R + f(Z_t^R)dt - ig(Z_t^R)dW_t - 2^{-1}g'(Z_t^R)g(Z_t^R)dt$$

Step 2: Prove that the H^1 norm remains finite
Introduce the Hamiltonian (recall that $f(Z) = F'(|Z|^2)Z$)

$$\int_M |\nabla Z(t,x)|^2 dx + \int_M F(|Z(t,x)|^2) dx$$

The deterministic conservation law shows that only the stochastic integral and Stratonovich-Itô correction term should be upper estimated (again Itô and Burkholder-Davies-Gundy)
Use the additional assumptions on the non-linearity $\tilde{g}$ satisfied if $\tilde{g} = 1$ (as in de Bouard Debussche) or e.g. if $\tilde{g}(r) = C_1 + C_2\sqrt{r+1}$

Work in progress

• Case $d = 3$
deterministic case proved by approximation of elements of H^1 by elements of H^s for $s > 1$ and if $F(r) = r^2$
Non linearity: $|z|^2 z$
• Case of a Dirac potential