HITTING PROBABILITIES FOR SYSTEMS OF STOCHASTIC WAVES

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Dedicated to the memory of
Paul Malliavin
(1925-2010)

Introduction

- $\{v(x), x \in \mathbb{R}^m\}$ is a $\mathbb{R}^d$-valued stochastic process.
- $I \subset \mathbb{R}^m$, compact, positive Lebesgue measure.
  Range of v:

$$v(I) = \{v(x), x \in I\}.$$

Question:

What can be said about $P\{v(I) \cap A \neq \emptyset\}$ in terms of $A \in \mathcal{B}(\mathbb{R}^d)$.

- Upper and lower bounds described by the capacity or the Hausdorff measure of A.
- Characterization of the polar sets A: $P\{v(I) \cap A \neq \emptyset\} = 0$.
- Hausdorff dimension of the range of v, questions on level sets, etc.

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Bessel-Riesz capacity

Bessel-Riesz kernel. For $\beta, r \in \mathbb{R}$,

$$K_\beta(r) = \begin{cases} r^{-\beta}, & \text{if } \beta > 0, \\ \log_+\left(\frac{1}{r}\right), & \text{if } \beta = 0, \\ 1, & \text{if } \beta < 0. \end{cases}$$

Energy. $E \in \mathcal{B}(\mathbb{R}^d)$, μ probability on E:

$$I_\beta(\mu) = \int_E \int_E K_\beta(\|x-y\|)\mu(dx)\mu(dy).$$

The capacity of E is:

$$\mathrm{Cap}_\beta(E) = \left[\inf_{\mu \in \mathcal{P}(E)} I_\beta(\mu)\right]^{-1}.$$

Hausdorff measure

For $\beta \in [0, \infty[$, $E \in \mathcal{B}(\mathbb{R}^d)$:

$$\mathcal{H}_\beta(E) = \liminf_{\varepsilon \to 0^+} \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : E \subset \cup_{i=1}^{\infty} B_{r_i}(x_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\}.$$

For $\beta \in]-\infty, 0[$, $E \in \mathcal{B}(\mathbb{R}^d)$:

$$\mathcal{H}_\beta(E) = \infty.$$

A useful fact relating capacities and Hausdorff measures

For $\beta_1 > \beta_2 > 0$ and compact E,

$$\mathrm{Cap}_{\beta_1}(E) > 0 \Longrightarrow \mathcal{H}_{\beta_1}(E) > 0 \Longrightarrow \mathrm{Cap}_{\beta_2}(E) > 0$$

(Frostman's theorem).

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Example 1: Brownian motion

For a d-dim Brownian motion B,

$$P(B(\mathbb{R}_+) \cap A \neq \emptyset) > 0 \iff \mathrm{Cap}_{d-2}(A) > 0.$$

In particular, for $x \neq 0$,

$$P(\exists t : B(t) = x) > 0 \iff d = 1.$$

Indeed,

$$\mathrm{Cap}_\beta(\{x\}) = \begin{cases} 1, & \beta < 0, \\ 0, & \beta \geq 0. \end{cases}$$

Kakutani, 1944; *Dvorestzky*, 1950.

For α–stable processes: McKean, 1955.

... And much more

- Markov processes: *Blumenthal and Getoor*, 1968.
- Multidimensional Markov processes: *Fitzsimmons and Salisbury*, 1989; *Hirsch and Song*, 1995.
- Lévy processes: *Bertoin*, 1996.
- Superprocesses: *Perkins*, 1990; *Dynkin*, 1991; *Le Gall*, 1994.
- Anisotropic Gaussian random fields: *Xiao*, 2007, 2008.
- ...

Some monographs:
J. L. Doob: *Classical Potential Theory and Its Probabilistic Counterpart*, 1984.
D. Khoshnevisan: *Multiparameter Processes: An Introduction to Random Fields*, 2002.

Example 2: Brownian sheet

$$\left\{W_{t_1,\dots,t_m} = \left(W^1_{t_1,\dots,t_m}, \dots, W^d_{t_1,\dots,t_m}\right), (t_1,\dots,t_m) \in \mathbb{R}^m_+\right\},$$

Gaussian, independent components, centered,

$$E\left(W^i_{t_1,\dots,t_m} W^i_{s_1,\dots,s_m}\right) = (t_1 \wedge s_1)\cdots(t_m \wedge s_m).$$

- *Khoshnevisan and Shi*, 1999:

$$c^{-1}\mathrm{Cap}_{d-2m}(A) \le P\{v(I) \cap A \neq \emptyset\} \le c\,\mathrm{Cap}_{d-2m}(A).$$

Remarks

- Extension of Kakutani's result ($m = 1$).
- Previous results on multidimensional Markov processes do not apply.

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Example 3: a system of SPDEs

$$\begin{cases} \partial^2_{t_1,t_2} u_i(t) = \sum_{j=1}^d \sigma^i_j(u(t)) \partial^2_{t_1,t_2} W^j_{t_1,t_2} + b^i(u(t)), & t = (t_1, t_2) \in \mathbb{R}^2_+, \\ u_i(t) = x_0, & t_1 t_2 = 0, \end{cases}$$

$i = 1, \dots, d$.

If $\sigma = \mathrm{Id_d}$, $b = 0$, then u is the Wiener sheet with $m = 2$.

- *Dalang and E. Nualart*, 2004: for $A \subset \mathbb{R}^d$, $I \subset \mathbb{R}^2$ compact sets,

$$c^{-1}\mathrm{Cap}_{d-4}(A) \leq P\{u(I) \cap A \neq \emptyset\} \leq c\,\mathrm{Cap}_{d-4}(A).$$

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Key ingredients for the extension

- Properties of the density of the solution $u(t)$ obtained using Malliavin calculus (*D. Nualart, S.-S*, 1985; *A. Kohatsu-Higa*, 2003).
- Two-parameter martingale inequalities (two-parameter stochastic calculus: *R. Cairoli, J.B. Walsh*, 1975).

In general, we may expect:

- Upper bounds in terms of Hausdorff measure.
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Results on systems of stochastic heat equations

- *Mueller, Tribe*, 2002: Additive, space-time white noise, spatial dimension $k = 1$, $b = 0$.
- *Dalang, Mueller, Zambotti*, 2006: SPDEs with reflection.
- *Dalang, Khoshnevisan, E. Nualart*, 2007: Additive, space-time white noise, spatial dimension $k = 1$.
- *Dalang, Khoshnevisan, E. Nualart*, 2009: Multiplicative noise, $k = 1$, space-time white noise.
- *Dalang, Khoshnevisan, E. Nualart* (*in progress*): Multiplicative noise, $k \geq 1$, white noise in t, correlated in space.

Criteria for Hitting Probabilities

R.C. Dalang, S.-S., 2009

Summary

- Lower bounds for hitting probabilities of A in terms of the Bessel-Riesz capacity of A are obtained assuming:
  1. Joint densities of $(v(x), v(y))$ have *Gaussian type* bounds.
  2. The density of $v(x)$ is bounded away from zero on compact sets.
- Upper bounds in terms of the Hausdorff measure are obtained supposing:
  1. The density of $v(x)$ is bounded above on compact sets.
  2. $E\left(\|v(x) - v(y)\|^q\right) \leq C\|x - y\|^{q\delta}$.

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Theorem 1: Lower bound

Assume:

1. For any $x, y \in \mathbb{R}^m$, $x \neq y$, $(v(x), v(y))$ has a density $p_{x,y}$, and there exist $\gamma, \alpha \in]0, \infty[$ such that

$$p_{x,y}(z_1, z_2) \leq C \frac{1}{\|x-y\|^{\gamma}} \exp\left(-\frac{\|z_1 - z_2\|^2}{\|x-y\|^{\alpha}}\right),$$

   for any $z_1, z_2 \in \mathbb{R}^d$.

2. The density p_x of $v(x)$ satisfies $\inf_{w \in K} p_x(w) > 0$, for any compact $K \subset \mathbb{R}^d$.

Then, for all $A \subset [-N, N]^d$, there exists $c > 0$ such that

$$P\{v(I) \cap A \neq \emptyset\} \geq c \mathrm{Cap}_{\frac{2}{\alpha}(\gamma - m)}(A).$$

Theorem 2: Upper bound

$D \subset \mathbb{R}^n$ is fixed. Assume

1. For any $x \in \mathbb{R}^m$, $v(x)$ has a density p_x, and

$$\sup_{x \in I, z \in D} p_x(z) \leq C.$$

2. There exists $\delta \in]0,1]$ such that for any $q \in [1, \infty[$, $x, y \in I$,

$$E\left(\|v(x) - v(y)\|^q\right) \leq C\|x - y\|^{q\delta}.$$

Then, for any $\theta \in]0, d[$ and any $A \subset D$,

$$P\{v(I) \cap A \neq \emptyset\} \leq C\mathcal{H}_{\theta - \frac{m}{\delta}}(A).$$

Remark: For some class of Gaussian processes, we can obtain

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Application to Systems of Stochastic Waves

A system of stochastic wave equations

$$\frac{\partial^2 u_i}{\partial t^2}(t,x) - \frac{\partial^2 u_i}{\partial x^2}(t,x) = \sum_{j=1}^{d} \sigma_{ij}(u(t,x))\dot{W}^j(t,x) + b_i(u(t,x)),$$

$1 \leq i \leq d$, $t \in [0, T]$, $x \in \mathbb{R}^k$, $k \geq 1$.

- $\dot{W} = (\dot{W}^1, \dots, \dot{W}^d)$ Gaussian noise, centered, with covariance

$$E\left(\dot{W}^i(t,x)\dot{W}^j(s,y)\right) = \delta(t-s)\|x-y\|^{-\beta}\delta_{ij},$$

$\beta \in (0, 2 \wedge k)$.

- b, σ Lipschitz continuous.

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References

- $k = 1$: *Cabaña*, 1970; *Walsh*, 1986; *Carmona and D. Nualart*, 1988.
- $k = 2$: *Dalang and Frangos*, 1998; *Dalang*, 1999, *Millet and S.-S*, 1999.
- $k = 3$: *Dalang*, 1999; *Peszat and Zabczyk*, 2000; *Dalang and S.S*, 2009.
- $k \geq 1$: *Peszat*, 2002, *Conus and Dalang*, 2008.
- *Ondreját*, 2004 – $\cdots$, *Brzeźniak*, $\ldots$

The Gaussian case

$\sigma = (\sigma_{ij})$ constant, $\det \sigma \neq 0$, $b_i = 0$, $i = 1, \dots, d$:

$$\frac{\partial^2 u_i}{\partial t^2}(t,x) - \frac{\partial^2 u_i}{\partial x^2}(t,x) = \sum_{j=1}^{d} \sigma_{ij} \dot{W}^j(t,x),$$

null initial conditions.
Solution:

$$u_i(t,x) = \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^k} G(t-r, x-y) \sigma_{ij} W^j(dr, dy)$$

$t \in [0, T]$, $x \in \mathbb{R}^k$, $k \geq 1$.

$G(t, \cdot)$ is the fundamental solution to the wave equation:

$$\mathcal{F} G(t, \cdot)(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|}.$$

Theorem 3: hitting probabilities for Gaussian waves

I, J are compact subsets of $[t_0, T]$ and $\mathbb{R}^k$, respectively, positive Lebesgue measure, $N > 0$, $t_0 > 0$. Then,

There exist positive constants $c_i = c_i(I, J, N, \beta, k, d)$, i=1,2, such that, for any Borel set $A \subset [-N, N]^d$,

$$c_1 \mathrm{Cap}_{d-\frac{2(k+1)}{2-\beta}}(A) \le P\{u(I \times J) \cap A \neq \emptyset\} \le c_2 \mathcal{H}_{d-\frac{2(k+1)}{2-\beta}}(A).$$

Theorem 3 (cont.): hitting probabilities for sections of Gaussian waves

1. For any $t \in I$, there exist positive constants $c_i = c_i(J, N, \beta, k, d)$, $i = 1, 2$, such that, for any Borel set $A \subset [-N, N]^d$,

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Related work for Gaussian processes: *Biermé, Lacaux, Xiao*, 2009.

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Application

When is a singleton $A = \{a\}$ a polar set for u?

- If A is *polar*, i.e. $P\{u(I \times J) \cap A \neq \emptyset\} = 0$, then

$$d - \frac{2(k+1)}{2-\beta} \geq 0.$$

- If $d - \frac{2(k+1)}{2-\beta} > 0$, then $\mathcal{H}_{d-\frac{2(k+1)}{2-\beta}}(A) = 0$; hence A is *polar*.

Conjecture: A is *polar* $\Leftrightarrow d - \frac{2(k+1)}{2-\beta} \geq 0$.

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Ingredients of the proof of Theorem 3

1. Bounds for the density

The density of $u(t,x)$ is $p_{t,x}(z) = \frac{1}{(2\pi\sigma_{t,x}^2)^{\frac{d}{2}}} \exp\left(-\frac{\|z\|^2}{2\sigma_{t,x}^2}\right)$, and

$$C(t \wedge t^3) \le \sigma_{t,x}^2 = \int_0^t ds \int_{\mathbb{R}^k} \frac{\sin^2(s\|\xi\|)}{\|\xi\|^2} \mu(d\xi) \le \tilde{C}(t + t^3),$$

$\mu(d\xi) = \|\xi\|^{\beta-k} d\xi.$
Hence,

$$\inf_{z\in[-N,N]^d} p_{t,x}(z) \ge C_1,$$

$$\sup_{z\in[-N,N]^d} \sup_{(t,x)\in[t_0,T]\times\mathbb{R}^k} p_{t,x}(z) \le C_2.$$

2. Estimates of the variance

For any $(t,x),(s,y) \in [t_0, T] \times \mathbb{R}^k$,

$$C_1 \left(|t-s| + \|x-y\|\right)^{2-\beta} \le E\left(\|u_{t,x} - u_{s,y}\|^2\right) \le C_2 \left(|t-s| + \|x-y\|\right)^{2-\beta}.$$

Why this is important?

1. L^p moments for increments.
2. By-product: regularity of the sample paths (optimal).
3. Improvement of the upper bound.
4. Upper bound for the joint density of $(u(t,x), u(s,y))$.

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3. Gaussian type upper bounds for joint densities

Let $p_{t,x;s,y}$ be the joint density of $(u(t,x),u(s,y))$. We have:

$$
\begin{aligned}
&p_{t,x;s,y}(z_1,z_2) = p_{t,x|s,y}(z_1,z_2)p_{s,y}(z_2) \\
&\le \frac{C}{(|t-s|+\|x-y\|)^{\frac{d(2-\beta)}{2}}} \exp\left(-\frac{\|z_1-z_2\|^2}{c\,(|t-s|+\|x-y\|)^{2-\beta}}\right).
\end{aligned}
$$

Hence

$$
\gamma = \frac{d(2-\beta)}{2}, \quad \alpha = 2-\beta,
$$

and $\frac{2}{\alpha}(\gamma - m) = d - \frac{2(k+1)}{2-\beta}$ (capacity dimension).

The general case

Assumptions (H)

- $k \in \{1, 2, 3\}$.
- $\sigma_{i,j}$, b_i, $1 \le i, j \le d$ bounded, infinitely differentiable, bounded partial derivatives of any order.
- σ is uniformly elliptic: for any $v \in \mathbb{R}^d$, $\|v\| = 1$,

$$\inf_{x \in \mathbb{R}^d} \|v^t \sigma(x)\| \ge \rho_0 > 0.$$

- $f(x) = \|x\|^{-\beta}$ if $x \neq 0$, with $\beta \in]0, 2 \wedge k[$.

Upper bound for the hitting probabilities of non-linear stochastic waves

Theorem 4 Assume (H).
I and K compact sets of $[0, T]$ and $\mathbb{R}^k$, respectively, positive Lebesgue measure. Fix $\delta \in (0,1)$.

There exists a positive constant $c = c(I, K, \beta, k, d, \delta)$ such that, for any Borel set $A \subset \mathbb{R}^d$,

$$P\{u(I \times K) \cap A \neq \emptyset\} \leq c\mathcal{H}_{d-\delta-\frac{2(k+1)}{2-\beta}}(A).$$

For sections:

$$P\{u(I \times \{x\}) \cap A \neq \emptyset\} \leq c\mathcal{H}_{d-\delta-\frac{2}{2-\beta}}(A).$$

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Ingredients

- Existence of density for the law of $u(t,x)$: $p_{t,x}(z)$.
- $\sup_{(t,x)\in K_1} \sup_{z\in K_2} p_{t,x}(z) \le C$.
- $E\left(\|u(t,x)-u(s,y)\|^p\right) \le C\left(|t-s|+\|x-y\|\right)^{\delta p}$, $\delta \in (0, \frac{2-\beta}{2})$.

Extensions of results from *Millet, S.-S., 99*, *Quer-Sardanyons, S.-S., 2004*, *Dalang, S.-S., 2009*.

Theorem 5: Lower bound for the hitting probabilities of non-linear stochastic waves

Assume (H). Let $I = [a, b] \subset (0, T]$, J compact subset of $\mathbb{R}^k$ of positive Lebesgue measure. Fix $\delta \in (0,1)$ and $N > 0$.
There exists a positive constant $c = c(I, J, N, \beta, k, d, \delta)$ such that, for any Borel set $A \subset [-N, N]^d$,

$$P\{u(I \times J) \cap A \neq \emptyset\} \geq c\mathrm{Cap}_{d\frac{\delta+3-\beta}{2-\beta} - \frac{2(k+1)}{2-\beta}}(A).$$

Theorem 5 (cont.): Lower bound for the hitting probabilities of sections of non-linear stochastic waves

1. Fix $\delta > 0$, $N > 0$ and $x \in J$. There exists a positive constant $c = c(I, N, \beta, k, d, x, \delta)$ such that, for any Borel set $A \subset [-N, N]^d$,

$$P\left\{u(I \times \{x\}) \cap A \neq \emptyset\right\} \geq c\mathrm{Cap}_{d\frac{\delta+3-\beta}{2-\beta} - \frac{2}{2-\beta}}.$$

2. Fix $\delta > 0$, $N > 0$ and $t \in I$. There exists a positive constant $c = c(J, N, \beta, k, d, t)$ such that, for any Borel set $A \subset [-N, N]^d$,

$$P\left\{u(\{t\} \times J) \cap A \neq \emptyset\right\} \geq c\mathrm{Cap}_{d\frac{\delta+3-\beta}{2-\beta} - \frac{2k}{2-\beta}}.$$

Remarks

- Upper bound is almost optimal
  - In the Gaussian case: $d - \frac{2(k+1)}{2-\beta}$.
  - In the non-Gaussian case: $d - \delta - \frac{2(k+1)}{2-\beta}$, δ arbitrarily small.
- Bessel-Riesz capacity dimension is not optimal.
  - In the Gaussian case: $d - \frac{2(k+1)}{2-\beta}$.
  - In the non-Gaussian case: $d\frac{\delta+3-\beta}{2-\beta} - \frac{2(k+1)}{2-\beta}$.

Ingredients

- Strict positivity of the density $p_{t,x}$ (extensions of work in *Aida-Kusuoka-Stroock*, 1991; *Bally-Pardoux*, 1998).
- Gaussian type bounds for the densities of $(u(s,y), u(t,x))$
  - For Gaussian waves

$$
\begin{aligned}
&p_{t,x;s,y}(z_1,z_2) = p_{t,x|s,y}(z_1,z_2)p_{s,y}(z_2) \\
&\leq \frac{C}{(|t-s|+\|x-y\|)^{\frac{d(2-\beta)}{2}}} \exp\left(-\frac{\|z_1-z_2\|^2}{c\,(|t-s|+\|x-y\|)^{2-\beta}}\right).
\end{aligned}
$$

  - For non-linear stochastic waves: Watanabe's formula for the density, based on the integration by parts formula of Malliavin calculus.

Formula for the density of $(u(s,y), u(t,x))$

$$
\begin{aligned}
p_{t,x;s,y}(z_1, z_2) &= \prod_{i=1}^{d} E\left(1_{\{|u_i(t,x)-u_i(s,y)|>|z_1^i - z_2^i|\}} H_{u(s,y),u(t,x)}\right) \\
&\leq \prod_{i=1}^{d} \left(P\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\}\right)^{\frac{1}{2d}} \\
&\quad \times \|H_{u(s,y),u(t,x)}\|_{L^2(\Omega)}.
\end{aligned}
$$

- $H_{u(s,y),u(t,x)}$ depends on:
  1. the Malliavin derivatives of $(u(s,y), u(t,x))$,
  2. the inverse of the Malliavin covariance matrix of $(u(s,y), u(t,x))$,
  3. $\cdots$

Contribution of $\|H_{u(s,y),u(t,x)}\|_{L^2(\Omega)}$

Theorem 6 Assume (H), then

$$\|H_{u(s,y),u(t,x)}\|_{L^2(\Omega)} \le \frac{C}{(|t-s| + \|x-y\|)^{\gamma}},$$

with $\gamma = \frac{d}{2}(\delta + 3 - \beta)$.

Remarks

- In the Gaussian case: $\gamma = \frac{d}{2}(2-\beta)$.
- Negative powers of $|t-s| + \|x-y\|$ appear in the estimates of the $L^p(\Omega)$-norm of the inverse of the Malliavin covariance matrix of $(u(s,y), u(t,x) - u(s,y))$.

Contribution of $\|H_{u(s,y),u(t,x)}\|_{L^2(\Omega)}$

Theorem 6 Assume (H), then

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Remarks

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Contribution of $P\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\}$

- $b = 0$

$$P\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\} \leq C_1 \exp\left\{-\frac{C_2|z_1^i - z_2^i|^2}{(|t-s| + \|x-y\|)^{2-\beta}}\right\}.$$

- $b \neq 0$

  No suitable Girsanov's type theorem

$$P\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\} \leq C\left[\frac{(|t-s| + \|x-y\|)^{\alpha}}{|z_1^i - z_2^i|} \wedge 1\right]^p,$$

with $\alpha \in \left(0, \frac{2-\beta}{2}\right)$, $p \geq 1$.

Contribution of $P\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\}$

- $b = 0$

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The End