Finite Difference Approximations of SPDEs

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joint work with Martin Hairer

Outline

Stochastic Burgers Equation

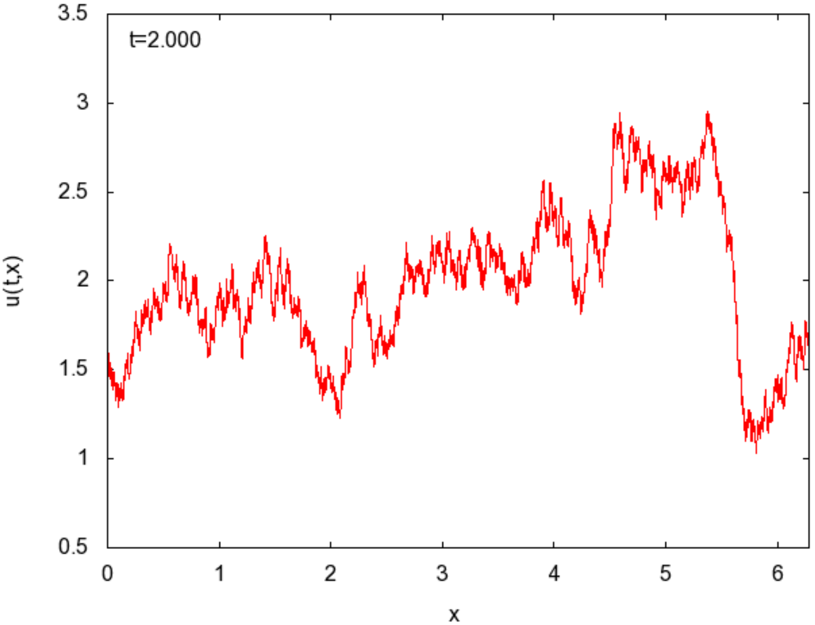
We consider the *stochastic Burgers equation*

$$du = \nu\,\partial_x^2 u\,dt - u\,\partial_x u\,dt + \sigma\,dw,$$

where $x \in [0, 2\pi]$, $t \geq 0$, the operator ∂_x^2 is equipped with periodic boundary conditions, and w is a cylindrical Wiener process.

- the $\partial_x^2 u$-term "smoothes" the solution
- the dw-term makes the solution "rough"
- $u\,\partial_x u$ is a transport term (shifts right if $u > 0$ and left if $u < 0$).

The (mild) solution satisfies $u(t) \in H^{1/2-\varepsilon}$ for all $\varepsilon > 0$.



Solution of the stochastic Burgers equation.

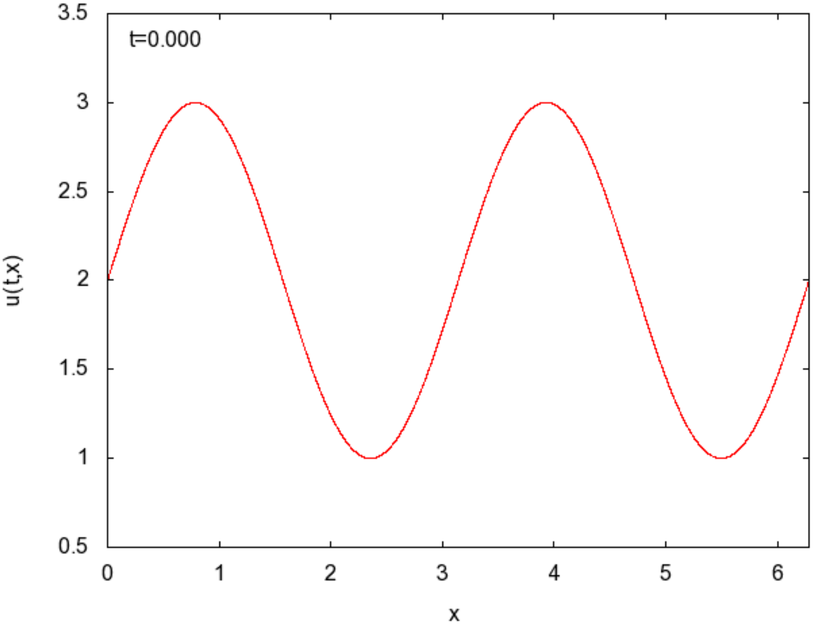
For $\nu = 0$, some care is needed when discretising this equation, even in the absence of noise:

- the solution can develop shocks
- when using finite difference discretisations, one needs to use an "upwind" scheme, *i.e.* to discretise $\partial_x u$ as

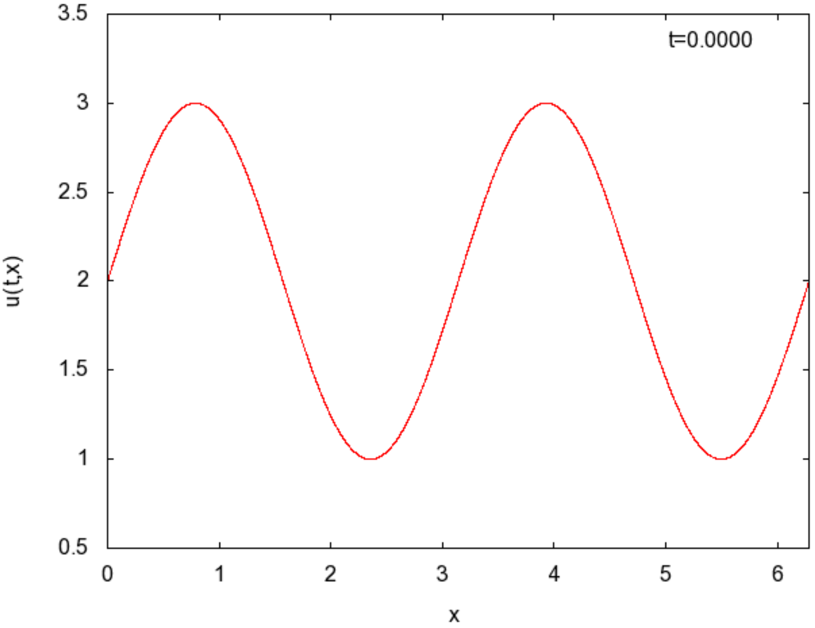
$$\frac{u(x) - u(x-\delta)}{\delta} \qquad \text{if } u(x) > 0$$

and

$$\frac{u(x+\delta) - u(x)}{\delta} \qquad \text{if } u(x) < 0.$$



Solution without viscosity and noise (*i.e.* $\nu = 0$, $\sigma = 0$).



Solution without viscosity and noise, “downwind discretisation”.

Finite Element Discretisation

We consider the approximating equation

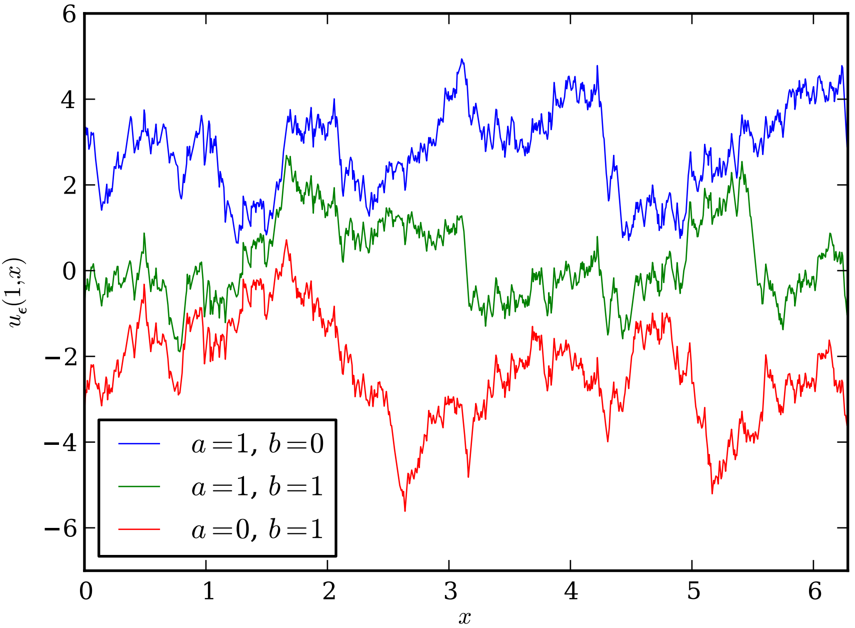
$$du = \nu\, \partial_x^2 u\, dt - u\, D_\delta u\, dt + \sigma\, dw,$$

where we define the approximate derivative D_δ by

$$D_\delta u(x) = \frac{u(x + a\delta) - u(x - b\delta)}{(a+b)\delta}$$

for some $a, b \geq 0$.

- In the absence of the noise term, this solution converges to the exact solution as $\delta \downarrow 0$.
- For the stochastic equation, this is not always the case (only for $a = b$).



Different discretisations of the stochastic Burgers equation.

$$D_\delta u(x) = \frac{u(x+a\delta) - u(x-b\delta)}{(a+b)\delta}$$

Conjecture 1. As $\delta \downarrow 0$, the solution of the approximating equation

$$du = \nu\, \partial_x^2 u\, dt - u D_\delta u\, dt + \sigma\, dw,$$

converges to the solution of

$$du = \nu\, \partial_x^2 u\, dt - u\, \partial_x u\, dt + \frac{\sigma^2}{4\nu} \frac{a-b}{a+b} + \sigma\, dw.$$

Remark. The problem can be avoided by discretising the nonlinearity $u\, \partial_x u = \frac{1}{2}\partial_x(u^2)$ as

$$\frac{u^2(x+a\delta) - u^2(x-b\delta)}{2(a+b)\delta}.$$

Heuristic derivation of the correction term.

Consider first the solution v to the stochastic heat equation

$$dv = \nu\,\partial_x^2 v\,dt + dw.$$

We would expect v to have the same smoothness properties as u.

The stationary solution is given by

$$v(t,x) = \sum_{n\neq 0} \frac{\xi_n(t)}{in\sqrt{2\nu}} \frac{\mathrm{e}^{inx}}{\sqrt{2\pi}} + B(t)\frac{1}{\sqrt{2\pi}},$$

where the ξ_n are complex-valued Ornstein-Uhlenbeck processes with $\mathbb{E}|\xi_n(t)|^2 = 1$ and time constant νn^2 that are independent, except for the condition that $\xi_{-n} = \bar{\xi}_n$.

The derivative of v is then (formally)

$$\partial_x v(x) = \sum_{n \neq 0} \frac{\xi_n(t) \mathrm{e}^{inx}}{2\sqrt{\nu\pi}}.$$

The δ-approximation to the derivative (as defined above) is

$$D_\delta v(x) = \sum_{n \neq 0} \frac{\xi_n(t) \mathrm{e}^{inx}}{2\sqrt{\nu\pi}} \frac{\mathrm{e}^{ina\delta} - \mathrm{e}^{-inb\delta}}{in(a+b)\delta}.$$

It is clear that the terms in the approximate derivative are a good approximation only up to $n \approx 1/\delta$.

Comparison of $v\,\partial_x v$ and $v\,D_\delta v$

- Since

$$\int_0^{2\pi} \frac{\mathrm{e}^{-i0x}}{\sqrt{2\pi}}\, v(x)\partial_x v(x)\,dx = \frac{1}{\sqrt{2\pi}}\Big(\frac{v^2}{2}(2\pi) - \frac{v^2}{2}(0)\Big) = 0,$$

  the $n = 0$ mode of $v\partial_x v$ vanishes.

- The $n = 0$ mode of $vD_\delta v$ can be found as

$$\sum_{k\neq 0} \frac{\xi_k(t)}{2\sqrt{\nu\pi}ik}\,\frac{\xi_{-k}(t)\big(\mathrm{e}^{-ika\delta} - \mathrm{e}^{ikb\delta}\big)}{2\sqrt{\nu\pi}i(-k)(a+b)\delta} = \sum_{k>0} \frac{|\xi_k(t)|^2}{2\pi\nu k}\,\frac{\cos kb\delta - \cos ka\delta}{(a+b)\delta k}$$

  which does not vanish in general ...

. . . Indeed, as $\delta \to 0$, the expectation of the 0-mode

$$\sum_{k>0} \frac{|\xi_k(t)|^2}{2\pi\nu k} \frac{\cos kb\delta - \cos ka\delta}{(a+b)\delta k}$$

converges to

$$\frac{1}{2\nu\pi} \int_0^\infty \frac{\cos bx - \cos ax}{(a+b)x^2}\, dx = \frac{1}{4\nu} \frac{b-a}{b+a},$$

which vanishes if and only if $a = b$.

More General Equations

Conjecture 2. As $\delta \to 0$, the solution of

$$du = \nu\, \partial_x^2 u\, dt + g(u) D_\delta u\, dt + \sigma\, dw$$

(where D_δ is the right-sided derivative) converges to the solution of

$$du = \nu\, \partial_x^2 u\, dt + g(u)\partial_x u\, dt - \frac{\sigma^2}{4\nu} g'(u)\, dt + \sigma\, dw. \tag{1}$$

Remarks.

- For $g(u) = -u$ this recovers conjecture 1.
- In the one-dimensional case we can always assume $g = h'$. Writing $g(u)\partial_x u = \partial_x\big(h(u)\big)$ one can easily see that the nonlinearity is regular enough for (1) to have a solution.
- In the multi-dimensional case, existence of solutions to (1) is no longer always clear.

Numerical evindence supporting conjectures 1 and 2.

We can test the conjectures using the following procedure.

1. For a small $\delta > 0$, compute the solution of

$$du_\delta = \nu\,\partial_x^2 u_\delta\,dt + g(u_\delta)D_\delta u_\delta\,dt + \sigma\,dw$$

   using right-sided discretisation and store the result.

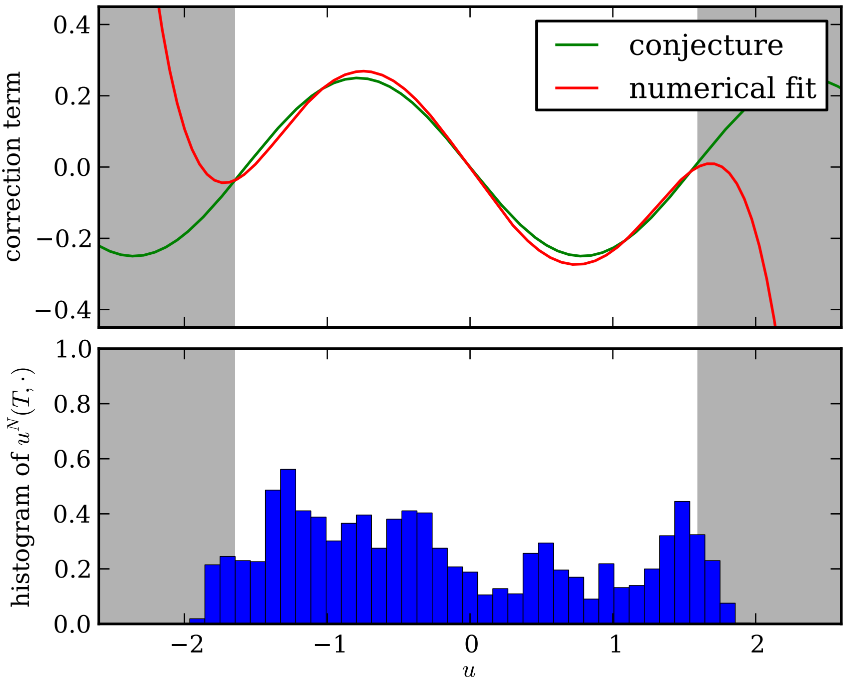
2. For the same instance of the noice, compute the solution of

$$du_\gamma = \nu\,\partial_x^2 u_\gamma\,dt + g(u_\gamma)\partial_x u_\gamma\,dt + \gamma(u_\gamma)\,dt + \sigma\,dw$$

   for different functions $\gamma\colon \mathbb{R} \to \mathbb{R}$, using a known-good discretisation.

3. vary γ to minimise $\left\| u_\delta - u_\gamma \right\|_2$.

If the conjectures are correct, we expect the minimum to be attained for $\gamma \approx -\frac{\sigma^2}{4\nu} g'$.



conjecture
numerical fit
correction term
0.4
0.2
0.0
−0.2
−0.4
1.0
0.8
0.6
0.4
0.2
0.0
histogram of $u^N(T,\cdot)$
−2
−1
0
1
2
u

Small Noise/Viscosity

Since the correction term in the conjectures above is proportional to σ^2/ν, we expect a non-vanishing correction term for arbitrarily small σ when taking $\nu \sim \sigma^2$.

In this section we consider discretisations of the equation

$$du = \varepsilon \, \partial_x^2 u \, dt - u \, \partial_x u \, dt + \sqrt{\varepsilon} \, dw.$$

We expect the following behaviour:

- as $\varepsilon \downarrow 0$ the solution converges to the viscosity solution of

$$\partial_t u = -u \, \partial_x u,$$

- as $\delta \downarrow 0$ the solution converges to the solution of

$$du = \varepsilon \, \partial_x^2 u \, dt - u \, \partial_x u \, dt + \frac{1}{4} + \sqrt{\varepsilon} \, dw.$$

Conjecture 3.

1. For $\delta \ll \varepsilon$, the right-sided discretisation converges to the viscosity solution of
$$\partial_t u = -u\,\partial_x u + \frac{1}{4}.$$
2. For $\varepsilon \ll \delta$, the right-sided discretisation converges to the viscosity solution of
$$\partial_t u = -u\,\partial_x u.$$

   Only upwind schemes will be stable.

Numerical evindence supporting conjecture 3.

Conjecture 3 can be tested using the following procedure:

1. Compute solutions of

$$\partial_t u_0 = -\frac{1}{2}\partial_x(u_0^2) \qquad \text{and} \qquad \partial_t u_1 = -\frac{1}{2}\partial_x(u_1^2) + \frac{1}{4},$$

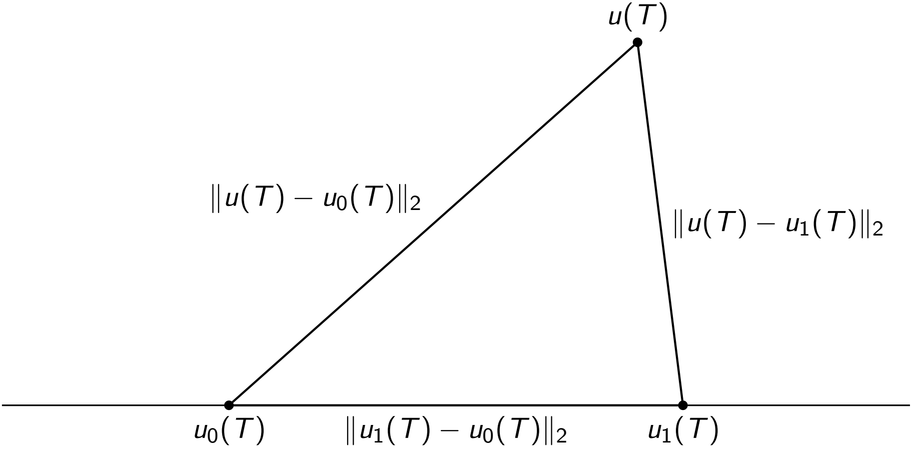
   using an upwind scheme.

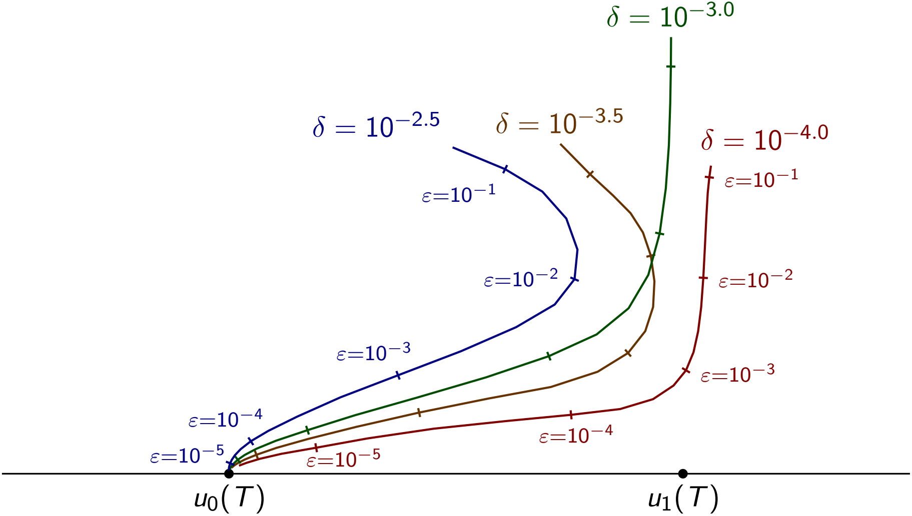
2. Given $\varepsilon, \delta > 0$, compute the right-sided discretisation of

$$du = \varepsilon\, \partial_x^2 u\, dt - u\, \partial_x u\, dt + \sqrt{\varepsilon}\, dw.$$

3. Compare the distances $\|u(T) - u_0(T)\|_2$ and $\|u(T) - u_1(T)\|_2$.

Given $\|u(T) - u_0(T)\|_2$ and $\|u(T) - u_1(T)\|_2$ we can map $u(T)$ into the two-dimensional plane:





$\delta = 10^{-2.5}$
$\delta = 10^{-3.0}$
$\delta = 10^{-3.5}$
$\delta = 10^{-4.0}$
$\varepsilon = 10^{-1}$
$\varepsilon = 10^{-2}$
$\varepsilon = 10^{-3}$
$\varepsilon = 10^{-4}$
$\varepsilon = 10^{-5}$
$\varepsilon = 10^{-1}$
$\varepsilon = 10^{-2}$
$\varepsilon = 10^{-3}$
$\varepsilon = 10^{-4}$
$\varepsilon = 10^{-5}$
$u_0(T)$
$u_1(T)$

Conclusion

- Finite difference discretisations can converge to the wrong solution! Some care is needed when discretising equations with "rough" noise.
- Using the heuristic method presented here, it is sometimes possible to guess the exact form of the error.
- The effect can be observed for arbitrarily small noise.