Effects of Three-Dimensional Geometry and Radial Electric Field on ITG Turbulence and Zonal Flows

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• Summary
Introduction
Large Helical Device (LHD)

Heliotron configuration
No net plasma current required
Suitable for steady-state operation

Max. parameters

\[ \begin{align*}
R &= 3.9 \text{ m} \\
a &= 0.6 \text{ m} \\
V &= 30 \text{ m}^3 \\
B &= 3 \sim 4 \text{ T} \\
n &= 1.1 \times 10^{21} \text{ m}^{-3} \\
T_e &= 15 \text{ keV} \\
T_H &= 5.2 \text{ keV} \\
\langle \beta \rangle &= 5.1 \% 
\end{align*} \]

1-hour discharge
Tokamak

\[ B = B_0 \left( 1 - \varepsilon_t \cos \theta \right) \]

Helical System

\[ B = B_0 \left[ 1 - \varepsilon_t \cos \theta - \varepsilon_h \cos (L\theta - M\zeta) \right] \]
Helical geometry influences ITG mode and zonal flow.

Eigenfunction of linear ITG mode electrostatic potential

Zonal-flow response (GAM, residual ZF)

Watanabe et al. NF2007

Sugama & Watanabe PoP2006
Gyrokinetic Equations (for ITG Turbulence) \[ k_{\perp} \rho_i \approx 1, \quad k_{\perp} \rho_e \ll 1 \]

**Ion gyrokinetic equation for** \( \delta f(x, v_\parallel, \mu, t) \)

\[
\begin{align*}
\frac{\partial}{\partial t} + v_\parallel \hat{b} \cdot \nabla + v_d \cdot \nabla - \mu \left( \hat{b} \cdot \nabla \Omega \right) \frac{\partial}{\partial v_\parallel} \delta f + \frac{e}{B_0} \left\{ \psi, \delta f \right\} &= \left( v_* - v_d - v_\parallel \hat{b} \right) \cdot \frac{e \nabla \psi}{T_i} F_M + C(\delta f)
\end{align*}
\]

- **Diamagnetic drift** \( v_* = -\frac{cT_i}{eL_n B_0} \left[ 1 + \eta_i \left( \frac{mv^2}{2T_i} - \frac{3}{2} \right) \right] \hat{y}, \quad \mu = \frac{v_{\perp}^2}{2\Omega} \)
- **Gyrocenter drift** \( v_d \cdot \nabla \)
- **Mirror force** \( -\mu (\hat{b} \cdot \nabla \Omega) \partial / \partial v_\parallel \rightarrow \quad \text{Effects of magnetic geometry} \)

**Quasineutrality condition & Adiabatic electron assumption**

\[
\int J_0(k_{\perp} v_\perp / \Omega) \delta f \, d^3 v - \left[ 1 - \Gamma_0(k_{\perp}^2) \right] e \frac{\phi}{T_i} = \frac{e}{T_e} \left( \phi - \langle \phi \rangle \right), \quad k_{\perp}^2 = \left( k_x + \hat{s}_z k_y \right)^2 + k_y^2
\]

**Ion polarization**
Linear ITG Mode Analysis for High-$T_i$ LHD plasmas
Fluctuation in High-$T_i$ discharge in LHD

K. Tanaka et al., to be appeared in Plasma Fusion Res.

$t = 1.833s$

$t = 2.233s$ (High $T_i$)

- Fluctuation peak exists at $\rho=0.8-1.0$ in space, $k_{p\rho_i} \sim 0.26$ in wavenumber.
- Fluctuation peak exists at $\rho=0.5-0.8$ in space, $k_{p\rho_i} \sim 0.45$ in wavenumber.
Results from Linear ITG Mode Analyses by GKV-X
(See Poster by M. Nunami)

Radial profiles of $\gamma_{\text{max}}$

- Growth rates are peaked at
  $\rho \sim 0.65$ (t=2.233s),
  $\rho \sim 0.85$ (t=1.833s).

Growth rates of ITG modes

- There exists ITG unstable region.
- Maximum growth rates exists at
  $k_{\theta}\rho_i \sim 0.35$ (t=2.233s),
  $k_{\theta}\rho_i \sim 0.20$ (t=1.833s),
  in poloidal wavenumber space.
Zonal Flows and ITG Turbulence
Gyrokinetic Simulation of EXB Zonal Flow Damping in Tokamaks

Undamped residual flow

\[ \phi_{k_r,0}(\infty) = \phi_{k_r,0}(0)/(1 + 1.6 q^2 / e^{1/2}) \]

After GAM oscillations are damped in the collisionless process (Landau damping), the zonal-flow potential approaches the theoretical value predicted by the Rosenbluth-Hinton theory.

Results from Gyrokinetic Vlasov (GKV) code

Structures of the perturbed gyrocenter distribution for zonal-flow components (tokamak case)

Simulation results

The gyrocenter distribution for residual zonal flow part can be described by the analytical solution.

\[
f_{k_x,0}(t) = F_M \frac{e^{\langle \phi_{k_x,0}(0) \rangle}}{T_i} \left[ k_x^2 \rho_i^2 + ik_x(\overline{\rho_b} - \rho_b) + k_x^2 (\rho_b \overline{\rho_b} - \frac{1}{2} \rho_b^2) - \frac{1}{2} \rho_b^2 \right] / (1 + 1.6q^2 / \epsilon^{1/2})
\]

Useful information to derive a kinetic-fluid closure model
Parallel heat fluxes:
\[
[q_\parallel, q_\perp] \equiv \int d^3v \, \delta f \left[ \left( m v_\parallel^2 - 3T \right) v_\parallel, \left( \frac{1}{2} m v_\perp^2 - T \right) v_\parallel \right]
\]

Fourth-order moments:
\[
[\delta r_{\parallel\parallel}, \delta r_{\parallel\perp}, \delta r_{\perp\perp}] \equiv \int d^3v \, \delta f \left[ m v_\parallel^4, \frac{1}{2} m v_\parallel^2 v_\perp^4, \frac{1}{4} m v_\perp^4 \right]
\]

\[
q = q^{(l)}_\parallel + q^{(s)}_\parallel
\]

\( (l) \) long-time behavior (residual zonal flow) + \( (s) \) short-time behavior (GAM damping)

Using the analytical solution \( \delta f \)

\[
q^{(l)}_{\parallel k_\perp} = -2q^{(l)}_{\perp k_\perp} = 2p_0 U_{k_\perp} \left[ B - \left( \beta_2 / \beta_1 \right) B^2 \right]
\]

\[
U_{k_\perp} = \beta_1 \left( \beta_1 - \langle B^{-2} \rangle \right)^{-1} \left[ \langle u_{\parallel k_\perp} / B \rangle - \langle B^{-2} \rangle \langle B u_{\parallel k_\perp} (t = 0) \rangle - (\beta_1 n_0)^{-1} \langle B^{-2} \rangle \left\langle \int d^3v \, F_0 R_{k_\perp} (t) (v_\parallel / B) \right\rangle \right].
\]

\[
\beta_1 = \frac{15}{4} \int_0^B d\lambda / \langle B / (1 - \lambda B)^{1/2} \rangle \quad R_{k_\perp} (t) = \int_0^t dt' S_{k_\perp} (t')
\]

\[
\beta_2 = \frac{3}{2} \int_0^B \lambda d\lambda / \langle B / (1 - \lambda B)^{1/2} \rangle
\]

different model from Beer & Hammett (1998)
Closure Model for Zonal Flow Dynamics in Tokamaks (II)

\[ q = q_{(l)} + q_{(s)} \]

(l) long-time behavior (residual zonal flow) + (s) short-time behavior (GAM damping)

\[
q_{(s)}^{(l)} = -2\sqrt{\frac{2}{\pi}} \, i n_0 v_t \sum_m \frac{m}{|m|} \delta T_{||m} e^{im\theta} \\
q_{(s)}^{(l)} = -\sqrt{\frac{2}{\pi}} \, i n_0 v_t \sum_m \frac{m}{|m|} \delta T_{\perp m} e^{im\theta}
\]

Hammett-Perkins type model

\[ n_0 \delta T_{||} = \delta p_{||} - T \delta n \]
\[ n_0 \delta T_{\perp} = \delta p_{\perp} - T \delta n \]

Fourth-order variables

\[ (\delta r_{||}, \delta r_{||\perp}, \delta r_{\perp\perp}) = (3, 1, 2) \times T v_t^2 \delta n^{(g)} \]

where the Maxwellian part of the perturbed distribution is taken into account.
ITG-Mode-Driven Zonal Flow in Tokamaks

Gyrofluid equations for ions combined with the quasineutrality condition

\[ e^{-\frac{b_i}{2}} \left( \frac{\delta n_{i\perp}^{(g)}}{n_0} - \frac{b_i}{2} \frac{\delta T_{i\perp}}{T_i} \right) - \frac{e\phi_{k\perp}}{T_i} [1 - \Gamma_0(b_i)] = \frac{e}{T_e} \left( \phi_{k\perp} - \langle \phi_{k\perp} \rangle \right) \]

Gyrofluid simulation shows a GAM damping process toward the same residual zonal-flow level as given by gyrokinetic simulation and the Rosenbluth-Hinton theory.

Rosenbluth-Hinton formula

\[ K_{R-H} = 1/(1 + 1.6q^2/\varepsilon_t^{1/2}) \]
ETG-Mode-Driven Zonal Flow

Gyrofluid equations for electrons combined with the Poisson equation

\[ e^{-b_e/2} \left( \frac{\delta n_{ek}^{(g)}}{n_0} - \frac{b_e}{2} \frac{\delta T_{e,k}}{T_e} \right) + \frac{e\phi_{k\perp}}{T_e} \left[ 1 - \Gamma_0(b_e + k_\perp \lambda_{De}^2) \right] = -\frac{e\phi_{k\perp}}{T_i} \]

Gyrofluid simulation shows the same residual zonal-flow level as given by gyrokinetic simulation and the analytical theory.

\[ \phi_{k\perp}(t) = \frac{T_e/T_i + k_\perp^2 (a_e^2 + \lambda_{De}^2)}{T_e/T_i + k_\perp^2 a_e^2 [1 + 1.6(1 + T_e/T_i)q^2/e^{1/2}] + k_\perp^2 \lambda_{De}^2} \phi_{k\perp}(0) \]

(b) \( k a_e = 0.172 \)
Collisionless Time Evolution of Zonal Flows in Helical Systems

[Sugama & Watanabe, PRL (2005), Phys.Plasmas (2006)]

Response of the zonal-flow potential to a given initial potential

$$\langle \phi_k (t) \rangle = K(t) \langle \phi_k (0) \rangle$$

Response function = GAM component + Residual component

$$K(t) = K_{GAM} (t) [1 - K_L (0)] + \left( \frac{K_L (t)}{1 - K_L (0)} \right)$$

$$K(t = 0) = 1 \quad K(t) \rightarrow K_L (t), \quad K_{GAM} (t) \rightarrow 0 \text{ as } t \rightarrow + \infty$$

GAM response function

$$K_{GAM} (t) = \cos(\omega_G) \exp(-|\gamma|t)$$

Long-time response function

$$\frac{1 - (2 / \pi)^{1/2} (2 \epsilon_H)^{1/2} \{1 - g_{1l} (t, \theta)\}}{1 + G + \frac{E(t)}{n_0 \langle k_{\perp}^2 \rho_i^2 \rangle}}$$

$$E(t)$$ represents effects of shielding of potential due to helical-ripple-trapped particles.

$$E(t) = \frac{2}{\pi} n_0 \frac{(2 \epsilon_H)^{1/2} \{1 - g_{1l} (t, \theta)\} - \frac{3}{2} \langle k_{\perp}^2 \rho_i^2 \rangle \langle (2 \epsilon_H)^{1/2} \{1 - g_{1l} (t, \theta)\} \rangle}{\langle (2 \epsilon_H)^{1/2} \{1 - g_{1l} (t, \theta)\} \rangle}$$

$$+ \frac{T_i}{T_e} \left( \frac{2 \epsilon_H}{(2 \epsilon_H)^{1/2} \{1 - g_{1l} (t, \theta)\}} \right)$$

$$B = B_0 \left[ 1 - \epsilon_i \cos \theta - \epsilon_h \cos (L \theta - M \zeta) \right]$$

$$\epsilon = 0.1$$

$L = 2, M = 10$
Perturbed gyrocenter distribution

\[ \delta f(v_\parallel, v_\perp) \]

Simulation

(a) \( \theta = 0 \)

(b) \( \theta = \frac{8\pi}{13} \)

Theory (rapid oscillations dropped)

(a) \( \theta = 0 \)

(b) \( \theta = \frac{8\pi}{13} \)

Helical plasma

\( q=1.5, \ \epsilon_h = 0.1, \ L=2, \ M=10 \)

\( t = 12.5 \ (R_0/v_t) \)
For low collisionality, better confinement is observed in the inward-shifted magnetic configurations, where lower neoclassical ripple transport but more unfavorable magnetic curvature driving pressure-gradient instabilities are anticipated.

Anomalous transport is also improved in the inward shifted configuration.

Scenario: Neoclassical optimization contributes to reduction of anomalous transport by enhancing the zonal-flow level.
Standard and Inward-shifted configurations

For the inward-shifted case, more unfavorable curvature but lower $q$ and higher magnetic shear $s$.

Larger residual zonal flow is found for the inward-shifted case.

The maximum ITG growth rate is slightly larger for the inward-shifted case.

Response of zonal-flow potential to a given initial potential

$\eta_i = L_n/L_{Ti} = 3$
$L_n/R_\theta = 0.3$
$T_e/T_i = 1$
Smaller $\chi_i$ and larger zonal flows are found in the saturated turbulent state for the inward-shifted configuration than for the standard one!

Larger residual zonal flow is found for the inward-shifted case.

Watanabe, Sugama & Ferrando, PRL(2008)
Sugama, Watanabe & Ferrando, PFR(2009)

The GKV turbulence simulations were carried out by the Earth Simulator (JAMSTEC).
Effects of Equilibrium Electric Field $E_r$ on Zonal Flows in Helical Systems

In helical systems $E_r$ is given from ambipolar condition of radial particle fluxes. $E_r$ reduces neoclassical ripple transport.

How does $E_r$ influence zonal flows and anomalous transport?
Effects of $E_r$ on gyrokinetic equation and zonal flows

Gyrokinetic equation for $k_\perp = k_r \nabla r$

\[
\frac{\partial}{\partial t} + v_\parallel \hat{b} \cdot \nabla + i k_r \cdot v_d - \mu (\hat{b} \cdot \nabla \Omega) \frac{\partial}{\partial v_\parallel} + \omega_E \frac{\partial}{\partial \alpha} \delta f = -i k_r \cdot v_d \frac{e \langle \phi(x + \rho) \rangle}{T_i} F_M
\]

angular velocity due to $\text{ExB}$ drift

\[\omega_E = -\frac{c E_r}{r_0 B_0}\]

field line label

\[\alpha = \theta - \zeta / q\]

In helical systems, $\alpha$ -dependence appears in $k_r \cdot v_d$ and $\mu (\hat{b} \cdot \nabla \Omega)$.

Therefore, even if the zonal-flow potential $\phi$ is independent of $\alpha$, $\delta f$ comes to depend on $\alpha$.

Thus, $\omega_E$ influences $\delta f$ and accordingly $\phi$ through quasineutrality condition.
Effects of Equilibrium $E_r$ on Zonal-Flow Response

Equilibrium $E_r$ field generates a ExB component to the velocity.

Poloidal ExB rotation of helically-trapped particles with reduced radial displacements $\Delta E$ will decrease the shielding of zonal-flow potential and increase its response.

Mynick & Boozer, PoP(2007)
Action-Angle Formulation
Classification of particle orbits in the presence of $E_r$

\[
\Delta_E \sim r_0 \frac{v_{dr}}{v_{E \times B}}
\]

radial displacement of helically-trapped particle

trapping parameter

\[
\kappa^2 = \frac{1 - \lambda B_0 [1 - \epsilon_T(\theta) - \epsilon_H(\theta)]}{2 \lambda B_0 \epsilon_H(\theta)}
\]

\[
\lambda = \frac{1}{2} \frac{mv^2}{\mu}
\]

Cary et al., PF (1988)
Wakatani (1998)
Solution of gyrokinetic equation to describe long-time evolution of zonal flows \cite{Sugama & Watanabe, PoP(2009)}

Perturbed particle distribution function

\[
\delta f_k(t) = -\frac{e}{T} \phi_k(t) F_M \left[ 1 - e^{-i k_r \rho_r} e^{-i k_r \Delta_r} \left\langle e^{i k_r \Delta_r} J_0(k_r \rho_r) \right\rangle_{\text{orbit}} \right] + e^{-i k_r \rho_r} e^{-i k_r \Delta_r} \left\langle e^{i k_r \Delta_r} \left[ \delta f_k^{(g)}(0) + F_M \int_0^t S_k(t) \, dt \right] \right\rangle_{\text{orbit}}
\]

Initial condition & Turbulence source

\[
P_t \approx \frac{4 \sqrt{2}}{\pi} \left( \frac{c E_r}{v B_0} \right) \left( \frac{R_0 q}{r_0} \right) \left( \epsilon_H \right)^{-1/2} \left[ \frac{\partial \epsilon_H}{\partial \theta} \right]_{\theta = \theta_0} (\epsilon_H - \epsilon_T)
\]

Polarization (classical & neoclassical)

Average along the orbit

\[
\langle \cdots \rangle_{\text{orbit}} = \frac{\int \cdots dl}{\int dl (dl / dt)}
\]

For particles which show transitions

\[
\frac{1}{\rho_r} \int \frac{dl}{(dl / dt)} = (1 - P_t) \int \frac{dl}{v_{\|} > 0} + \int \frac{dl}{v_{\|} < 0} + P_t \int \frac{d\theta}{\omega_E}
\]

\( P_t \) transition probability

ExB drift

\( P_t \) 1-

1

1

\( P_t \) transition probability

For particles which show transitions

\[
\frac{1}{\Delta_r} \int \frac{dl}{(dl / dt)} = \int \frac{dl}{v_{\|} > 0} + \int \frac{dl}{v_{\|} < 0} + \int \frac{d\theta}{\omega_E}
\]

\( P_t \) transition probability

For particles which show transitions

\[
\frac{1}{\Delta_r} \int \frac{dl}{(dl / dt)} = \int \frac{dl}{v_{\|} > 0} + \int \frac{dl}{v_{\|} < 0} + \int \frac{d\theta}{\omega_E}
\]
Long-time zonal-flow response to the initial condition and turbulence source

For radial wavenumbers $k_r \rho_{ti} < 1$ (ITG turbulence) and $k_r \Delta_E < 1$, the zonal-flow potential is derived from the quasineutrality condition as [Sugama & Watanabe, PoP(2009)]

$$
\frac{e}{T_i} \langle \phi_k(t) \rangle = \left\langle n_0^{-1} \int d^3v \left[ 1 + i k_r \left( \Delta_{ir} - \left\langle \Delta_{ir} \right\rangle_{\text{orbit}} \right) \right] \left[ \delta f_{ik}^{(g)}(0) + F_M \int_0^t S_{ik}(t) \, dt \right] \right\rangle.
$$

Geometrical factors $G$’s represents shielding effects of neoclassical polarization due to particles motions in different orbits.

$G \propto \text{(population)} \times \left( \Delta_r / \rho \right)^2$

- $G_p$: passing
- $G_t$: toroidally-trapped
- $G_h$: helically-trapped (unclosed orbit)
- $G_{ht}$: toroidally-trapped (closed orbit)

Zonal-flow generation can be enhanced when $G_{ht}$ and $G_h$ decreases with neoclassical optimization (which reduces radial drift velocity $V_{dr}$) and when poloidal Mach number $M_p \equiv \left| (cE_r / B_0) / (r v_{ti} / Rq) \right|$ increases with increasing $E_r$ and using heavier ions.
Response to the initial condition

Assume the initial distribution to have Maxwellian dependence

\[ \delta f^{(g)}_{ik}(0) = \frac{\delta n^{(g)}_k(0)}{n_0} F_M \]

Then, we obtain

\[ \delta n^{(g)}_k(0)/n_0 = (k_r \rho_n)^2 e \phi(0)/T_i \]

\[ \langle \phi_k(t) \rangle = \frac{\langle \phi_k(0) \rangle}{1 + G_p + G_t + M_p^{-2}(G_{ht} + G_h)(1 + T_e/T_i)} \]

(no turbulence source)

For the single-helicity configuration

\[ B = B_0[1 - \varepsilon_i \cos \theta - \varepsilon_h \cos(L \theta - M \xi)] \quad (\varepsilon_h : \text{independent of } \theta) \]

No transitions occur.

\[ G_{ht} = 0, \quad G_h = (15 \pi / 4) q^2 (2\varepsilon_h)^{1/2} \]

\[ \langle \phi_k(t) \rangle = \frac{\langle \phi_k(0) \rangle}{1 + G_p + G_t + (15 \pi / 4) M_p^{-2} q^2 (2\varepsilon_h)^{1/2}(1 + T_e/T_i)} \]

This corresponds to the case considered by previous works.

Mynick & Boozer, PoP(2007)
Sugama, Watanabe & Ferrando, PFR(2008)
Extention of GKV code to poloidally global model

GKV code is extended from the flux tube to the poloidally global model for studying effects of $E_r$ on zonal flows in helical systems [Watanabe, IAEA FEC 2008].

$$\alpha \equiv \theta - \zeta / q : \text{field-line label} \quad \zeta : \text{toroidal angle}$$

- Linear simulations for time evolution of zonal flows are done using
  - 129 Fourier modes in the $\alpha$ direction ,
  - 1,536 grid points in the $\zeta$ direction, and
  - (512, 48) grid points in the $(v_{||}, \mu)$ space for a fixed radial wavenumber $k_r$.

- Standard configuration model (single helicity):
  $$B = B_0[1 - \varepsilon_t \cos \theta - \varepsilon_h \cos(L\theta - M\zeta)], \quad \varepsilon_t = 0.1, \quad \varepsilon_h = 0.1, \quad q = 1.5, \quad L = 2, \quad M = 10$$

- Inward-shifted configuration model:
  Sideband helicity components \( \varepsilon_{L+1} = -0.02, \varepsilon_{L-1} = -0.08 \) are included.
Collisionless time evolution of zonal flows in helical configurations with $E_r$.

It is clearly shown for the inward-shifted model configuration that the residual zonal-flow potential amplitude (observed after Landau damping of GAM) is enhanced by increasing $E_r$.

**Inward-Shifted Model**

$$\frac{\phi(t)}{\phi(0)}$$

**Single Helicity**

$$\frac{\phi(t)}{\phi(0)}$$

- $M_p = 0$
- $M_p = 0.1$
- $M_p = 0.2$
- $M_p = 0.3$

(k_r $\rho_i = 0.131$)
The residual zonal-flow potential as a function of $k_r \rho_{ti}$ for $M_p = 0$ and $M_p = 0.3$

Different $k_r \rho_{ti}$ dependences for $M_p = 0$ and $M_p = 0.3$ are theoretically predicted and confirmed by simulation.

Theoretical results are derived by assuming $k_r \rho_{ti} \ll 1$.

[To be published in CPP]
Dependence of the residual zonal-flow potential on the poloidal Mach Number \((M_p)\) for \(k_r \rho_{ti} = 0.065\)

Residual zonal-flow potential increases with increasing \(M_p\).

Qualitative agreement between theory and simulation is verified.

More details are found in poster by T.-H. Watanabe [submitted to PPCF]
Momentum Balance and Radial Electric Field in Quasisymmetric Systems with Stellarator Symmetry
Basic Boltzmann Kinetic Equation for description of Collisional and Turbulent Transport

Equilibrium magnetic field

\[ \mathbf{B} = \psi' \nabla s \times \nabla \theta + \chi' \nabla \zeta \times \nabla s = B_s \nabla s + B_\theta \nabla \theta + B_\zeta \nabla \zeta \]

Boltzmann kinetic equation

\[ \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{e_a}{m_a} \left\{ (\mathbf{E} + \hat{\mathbf{E}}) + \frac{1}{c} \mathbf{v} \times (\mathbf{B} + \hat{\mathbf{B}}) \right\} \cdot \frac{\partial}{\partial \mathbf{v}} \right] (f_a + \hat{f}_a) = C_a (f_a + \hat{f}_a) \]

Ensemble-averaged kinetic equation

\[ \frac{d}{dt} f_a \equiv \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{e_a}{m_a} \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}} \right] f_a = \langle C_a \rangle_{\text{ens}} + D_a \]

\[ D_a = -\frac{e_a}{m_a} \left\langle \left( \hat{\mathbf{E}} + \frac{1}{c} \mathbf{v} \times \hat{\mathbf{B}} \right) \cdot \frac{\partial \hat{f}_a}{\partial \mathbf{v}} \right\rangle_{\text{ens}} \]
The gyrophase ($\xi$) -average part and the oscillating part of an arbitrary function $F$ is defined by $\bar{F} \equiv (2\pi)^{-1} \oint d\xi \, F$ and $\tilde{F} \equiv F - \bar{F}$ respectively.

Particle flux
$$\Gamma_a \equiv \langle \Gamma_a \cdot \nabla s \rangle \equiv \left\langle \int d^3v \, \tilde{f}_a v \cdot \nabla s \right\rangle$$

Heat flux
$$\frac{q_a}{T_a} \equiv \frac{\langle q_a \cdot \nabla s \rangle}{T_a} \equiv \left\langle \int d^3v \, \tilde{f}_a \left( \frac{m_a v^2}{2T_a} - \frac{5}{2} \right) v \cdot \nabla s \right\rangle$$

The ensemble-averaged kinetic equation is divided as
$$\overline{\mathcal{L}(\tilde{f}_a + \bar{f}_a)} = \overline{\langle C_a \rangle}_{\text{ens}} + \overline{\mathcal{D}_a}, \quad \Omega_a \frac{\partial \tilde{f}_a}{\partial \xi} = \mathcal{L} \tilde{f}_a - \overline{\langle C_a \rangle}_{\text{ens}} - \overline{\mathcal{D}_a}$$

Second order part of $\tilde{f}_a$ in $\delta \sim \rho / L$
$$\tilde{f}_{a2} = \tilde{f}_a^N + \tilde{f}_a^H + \tilde{f}_a^C + \tilde{f}_a^A \equiv \frac{1}{\Omega_a} \int_{-\xi}^{\xi} d\xi \left[ \mathcal{L} \tilde{f}_{a1} + \tilde{\mathcal{L}} \tilde{f}_{a1} - C_a^L(\tilde{f}_{a1}) - \overline{\mathcal{D}_a} \right]$$

$$\Gamma_a = \Gamma_a^{\text{ncl}} + \Gamma_a^{\text{cl}} + \Gamma_a^{\text{anom}} \quad q_a = q_a^{\text{ncl}} + q_a^{\text{cl}} + q_a^{\text{anom}}$$
Momentum Balance

\[
\frac{\partial}{\partial t}(n_a m_a u_a) = -\nabla \cdot P_a + n_a e_a \left( E + \frac{u_a}{c} \times B \right) + F_{a1} + K_{a1}
\]

density \quad n_a \equiv \int d^3v \ f_a \quad \text{particle flux} \quad n_a u_a \equiv \int d^3v \ f_a v

pressure tensor \quad P_a \equiv \int d^3v \ f_a m_a v v

friction force \quad F_{a1} \equiv \int d^3v \ C_a(f_a) m_a v

turbulent electromagnetic force \quad K_{a1} \equiv \int d^3v \ D_a v

\[
\sum_a K_{a1} = \nabla \cdot \left\langle \frac{1}{4\pi} \left( \hat{E} \hat{E} + \hat{B} \hat{B} \right) - \frac{1}{8\pi} \left( \hat{E}^2 + \hat{B}^2 \right) I \right\rangle_{\text{ens}} - \frac{1}{4\pi c} \frac{\partial}{\partial t} \left\langle \hat{E} \times \hat{B} \right\rangle_{\text{ens}}
\]

\[= \nabla \cdot T_{EM} - \frac{\partial}{\partial t} \left( \frac{S_{EM}}{c^2} \right),\]
Momentum Balance in the direction tangential to the flux surface

\[
\frac{\partial}{\partial t} \sum_a n_a m_a \left\{ c_1 \left( u_{a\theta} + \frac{(S_{EM})_{\theta}}{c^2} \right) + c_2 \left( u_{a\zeta} + \frac{(S_{EM})_{\zeta}}{c^2} \right) \right\} = -\frac{1}{V'} \frac{\partial}{\partial s} \left[ V' \left\langle \nabla_s \cdot \left( \sum_a P_a - T_{EM} \right) \cdot \left( c_1 \frac{\partial \mathbf{x}}{\partial \theta} + c_2 \frac{\partial \mathbf{x}}{\partial \zeta} \right) \right\rangle \right] + \frac{1}{c} \left( -c_1 \psi' + c_2 \chi' \right) \sum_a e_a \left\langle n_a u_a^s \right\rangle
\]

\( (s, \theta, \zeta) : \text{Hamada coordinates} \)

The surface-averaged radial current

\[
\sum_a e_a \Gamma_a \equiv \sum_a e_a \left\langle n_a u_a^s \right\rangle = -\frac{1}{4\pi} \frac{\partial}{\partial t} \left\langle E_a^s \right\rangle
\]
Quasisymmetry


\[ c_1 \frac{\partial B}{\partial \theta} + c_2 \frac{\partial B}{\partial \zeta} = 0 \]

quasi-axi-symmetry \((c_1, c_2) = (0, 1)\)

quasi-poloidal-symmetry \((c_1, c_2) = (1, 0)\)

The \(O(\delta)\) viscosity component in the quasisymmetry direction vanishes:

\[
\left\langle \left( c_1 \frac{\partial x}{\partial \theta} + c_2 \frac{\partial x}{\partial \zeta} \right) \cdot \left[ \nabla \cdot \left\{ P_{||a} bb + P_{\perp a} (I - bb) \right\} \right] \right\rangle \\
= - \left\langle \left( P_{||a} - P_{\perp a} \right) \left( c_1 \frac{\partial x}{\partial \theta} + c_2 \frac{\partial x}{\partial \zeta} \right) \cdot \nabla \ln B \right\rangle = 0
\]

The ambipolarity \(\sum_a e_a \langle n_a u_a^s \rangle = 0\) is satisfied automatically up to \(O(\delta)\).
Stellarator Symmetry

Magnetic field strength

\[ B(s, -\theta, -\zeta) = B(s, \theta, \zeta) \]

Magnetic field components

\[ B^\theta(s, -\theta, -\zeta) = B^\theta(s, \theta, \zeta), \quad B^\zeta(s, -\theta, -\zeta) = B^\zeta(s, \theta, \zeta) \]
\[ B_\theta(s, -\theta, -\zeta) = B_\theta(s, \theta, \zeta), \quad B_\zeta(s, -\theta, -\zeta) = B_\zeta(s, \theta, \zeta) \]
\[ B_s(s, -\theta, -\zeta) = -B_s(s, \theta, \zeta), \]

Metric tensor components

\[ g_{ss}(s, -\theta, -\zeta) = g_{ss}(s, \theta, \zeta), \quad g_{\theta\theta}(s, -\theta, -\zeta) = g_{\theta\theta}(s, \theta, \zeta) \]
\[ g_{\theta\zeta}(s, -\theta, -\zeta) = g_{\theta\zeta}(s, \theta, \zeta), \quad g_{\zeta\zeta}(s, -\theta, -\zeta) = g_{\zeta\zeta}(s, \theta, \zeta) \]
\[ g_{s\theta}(s, -\theta, -\zeta) = -g_{s\theta}(s, \theta, \zeta), \quad g_{s\zeta}(s, -\theta, -\zeta) = -g_{s\zeta}(s, \theta, \zeta) \]
\[ g(s, -\theta, -\zeta) = g(s, \theta, \zeta), \]
Parity Transformation associated with Stellarator Symmetry

Expansion in $\eta \sim \delta \sim \rho / L$ (Put $e_a \rightarrow \eta^{-1}e_a$ in Boltzmann and Maxwell eqs.)

$f_a(s, \theta, \zeta, v^s, v^\theta, v^\zeta, t, \eta) = f_{aM}(s, v, \eta^2t) + \eta f_{a1}(s, \theta, \zeta, v^s, v^\theta, v^\zeta, \eta^2t) + \eta^2 f_{a2}(s, \theta, \zeta, v^s, v^\theta, v^\zeta, \eta^2t) + \cdots,$

$\Phi(s, \theta, \zeta, t, \eta) = \eta \Phi_1(s, \eta^2t) + \eta^2 \Phi_2(s, \theta, \zeta, \eta^2t)$

Parity operator $\mathcal{P}$ is defined by

$(\mathcal{P}Q)(s, \theta, \zeta, v^s, v^\theta, v^\zeta, t, \eta) \equiv Q(s, -\theta, -\zeta, v^s, -v^\theta, -v^\zeta, t, -\eta)$

In the presence of stellarator symmetry, Boltzmann and Maxwell equations are invariant under parity transformation

$f_a + \hat{f}_a \quad \longrightarrow \quad \mathcal{P}(f_a + \hat{f}_a)$

$E_s + \hat{E}_s$, $E_\theta + \hat{E}_\theta$, $E_\zeta + \hat{E}_\zeta \quad \longrightarrow \quad -\mathcal{P}(E_s + \hat{E}_s)$, $\mathcal{P}(E_\theta + \hat{E}_\theta)$, $\mathcal{P}(E_\zeta + \hat{E}_\zeta)$

$B_s + \hat{B}_s$, $B_\theta + \hat{B}_\theta$, $B_\zeta + \hat{B}_\zeta \quad \longrightarrow \quad -\mathcal{P}(B_s + \hat{B}_s)$, $\mathcal{P}(B_\theta + \hat{B}_\theta)$, $\mathcal{P}(B_\zeta + \hat{B}_\zeta)$
Momentum Transport Fluxes in Stellarator Symmetric Systems

Parity of solutions

\[ \mathcal{P} f_a = f_a, \quad -\mathcal{P} \Phi = \Phi \]
\[ f_j(s, -\theta, -\zeta, v^s, -v^\theta, -v^\zeta, \eta^2 t) = (-1)^j f_j(s, \theta, \zeta, v^s, v^\theta, v^\zeta, \eta^2 t) \]
\[ \Phi_j(s, -\theta, -\zeta, \eta^2 t) = (-1)^{j-1} \Phi_j(s, \theta, \zeta, \eta^2 t) \]

When \( j \) is even, the \( O(\delta^j) \) part of radial transport fluxes of poloidal and toroidal momentum vanish.

\[ \left\langle \left( P_a^{(j)} \right)^s \right\rangle_{\theta} = \left\langle \left( P_a^{(j)} \right)^s \right\rangle_{\zeta} = \left\langle \left( T_{EM}^{(j)} \right)^s \right\rangle_{\theta} = \left\langle \left( T_{EM}^{(j)} \right)^s \right\rangle_{\zeta} = 0 \quad (\text{for even } j) \]
Momentum Balance in Quasisymmetric Systems with Stellarator Symmetry

In quasisymmetric systems with stellarator symmetry, the momentum transport fluxes vanish up to $O(\delta^2)$, and the ambipolarity is automatically satisfied up to $O(\delta^2)$.

The momentum balance equation determining $E_s$ is of $O(\delta^3)$:

$$\frac{\partial}{\partial t} \left[ \frac{(c_2\chi' - c_1\psi')}{4\pi c} \left\{ \left| \nabla_s \right|^2 + \frac{4\pi c^2 \sum_a n_am_a}{(c_2\chi' - c_1\psi')^2} \left( \left| c_1 \frac{\partial x}{\partial \theta} + c_2 \frac{\partial x}{\partial \zeta} \right|^2 \right) \right\} E_s \right]$$

$$+ \sum_a \frac{m_a}{(c_2\chi' - c_1\psi')} \left\{ -\frac{c}{e_a} \frac{\partial p_a}{\partial s} \left( \left| c_1 \frac{\partial x}{\partial \theta} + c_2 \frac{\partial x}{\partial \zeta} \right|^2 \right) + \frac{n_a V'}{4\pi^2} \left( c_1 B_\theta + c_2 B_\zeta \right) \left( c_2 u_\theta - c_1 u_\zeta \right) \right\}$$

$$= -\frac{1}{V'} \frac{\partial}{\partial s} \left[ V' \left( \nabla_s \cdot \left( \sum_a \mathbf{P}_a^{(3)} - \mathbf{T}_{EM}^{(3)} \right) \right) \cdot \left( c_1 \frac{\partial x}{\partial \theta} + c_2 \frac{\partial x}{\partial \zeta} \right) \right]$$
Momentum Balance in Toroidally Rotating Tokamaks
with Toroidal Velocity $V \sim v_{Ti}$  [Sugama& Horton, PoP1998]

Toroidal flow is proportional to the radial electric field

$$V_0 = RV^\xi = -Rc \frac{\partial \Phi_0(\Psi)}{\partial \Psi}$$

The momentum balance equation determining $E_s$ is of $O(\delta^2)$:

$$\frac{\partial}{\partial t} \left( \sum_a m_a n_a \left( 1 + \frac{v_{PA}^2}{c^2} \right) R^2 V^\xi \right) + \frac{1}{V'} \frac{\partial}{\partial \Psi} \left( V' \sum_a \Pi_a \right) = \sum_a \left( \int d^3v \ m_a v^\xi (D_a + I_a) \right)$$

Toroidal momentum flux is of $O(\delta^2)$.

$$\Pi_a = \Pi_{a}^{cl} + \Pi_{a}^{ncl} + \Pi_{a}^{H} + \Pi_{a}^{(E)} + \Pi_{a}^{anom}$$
Neoclassical and Anomalous Toroidal Momentum Fluxes in Toroidally Rotating Tokamaks

Neoclassical toroidal momentum flux of $O(\delta^2)$. 

$$\Pi^{nc1}_a + \Pi^H_a = -m_a c IV^\xi (n_a R^2) \frac{\langle BE^{(A)} \rangle}{\langle B^2 \rangle} - \frac{m_a c}{2e_a} \int d^3v \left[ m_a \left( R^2 V^\xi + \frac{I}{B} v'_|| \right) + \mu \frac{R^2 B^2_p}{B} C^L_a (g_a) \right]$$

$$v'_|| b \cdot \nabla g_a - C^L_a (g_a) = \frac{1}{T} f_{a0} (W_{a1} X_{a1} + W_{a2} X_{a2} + W_{av} X_v + W_{av} X_E)$$

$$X_v = -\frac{\partial V^\xi}{\partial \psi} = c \frac{\partial^2 \Phi_0}{\partial \psi^2}$$

$$W_{av} = \frac{m_a c}{2e_a} v'_|| b \cdot \nabla \left[ m_a \left( R^2 V^\xi + \frac{I}{B} v'_|| \right)^2 + \mu \frac{R^2 B^2_p}{B} \right]$$

Anomalous toroidal momentum flux of $O(\delta^2)$. 

$$\Pi^A_a = \left\langle \left\langle \int d^3v \hat{h}_a(X) \hat{w}_{av}(X) \right\rangle \right\rangle$$

$$\hat{w}_{av}(X) = \left\langle -\frac{c}{B} \nabla \left( \phi(x) - \frac{1}{c} (V_0 + v') \cdot \hat{A}(x) \right) \times b \cdot \nabla \psi m_a (V_0 + v') \cdot (R \hat{\xi}) \right\rangle_X$$

$$\left[ \frac{\partial}{\partial t} + \left( V_0 + v'_|| b + v_{da} - \frac{c}{B} \nabla \hat{\psi}_a(X) \times b \right) \cdot \nabla \right] \hat{h}_a(X) - \langle C^L_a [\hat{f}_a(X + \rho_a)] \rangle_X$$

$$= \frac{c}{B} \nabla \hat{\psi}_a(X) \times b \left[ \nabla + \left\{ \frac{e_a}{T_a} \frac{\partial \langle \Phi_1 \rangle}{\partial \psi} + \frac{m_a}{T_a} \left( R^2 V^\xi + \frac{I}{B} v'_|| \right) \frac{\partial V^\xi}{\partial \psi} \right] \nabla \psi \right\} f_{a0} + \frac{e_a}{T_a} \left[ \left( \frac{\partial}{\partial t} + V_0 \cdot \nabla \right) \hat{w}_a(X) \right] f_{a0}$$
Quasi-axisymmetric System with Toroidal Velocity $V \sim v_{Ti}$

Toroidal flow

$$V = V_\zeta \frac{\partial x}{\partial \zeta}, \quad V_\zeta = -c \frac{\Phi_0(s)}{\chi'(s)} = \mathcal{O}(v_{Ti})$$

Equilibrium force balance

$$\left( \sum_a n_a m_a \right) V \cdot \nabla V = \frac{1}{c} J \times B - \nabla P$$

Toroidal component

$$\frac{1}{2} \left( \sum_a n_a m_a \right) (V_\zeta)^2 \frac{\partial g_{\zeta\zeta}}{\partial \zeta} = \frac{\chi'}{c} J^s = \frac{B^\theta}{c} \left( \frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta} \right)$$

Generally, \( \frac{\partial g_{\zeta\zeta}}{\partial \zeta} \neq 0 \) Therefore, \( J^s \neq 0 \)

Then, neither Boozer nor Hamada coordinates can be constructed. Thus, high toroidal velocity on the order of ion thermal velocity does not seem to be allowed by simple quasiaxisymmetry condition only.
Summary

- Fluctuations observed in a high $T_i$ LHD plasma are considered as ITG modes predicted from linear calculation by GKV-X.

- Zonal-flow response theory and simulation show that zonal flow generation and turbulence regulation are enhanced when the radial displacements of helical-ripple-trapped particles are reduced either by neoclassical optimization of the helical geometry lowering the radial drift velocity or by strengthening the radial electric field $E_r$ to boost the poloidal rotation.

- The $E_r$ effects appear through the poloidal Mach number $M_p$. For the same magnitude of $E_r$, higher zonal-flow response is obtained by using ions with heavier mass (favorable deviation from gyro-Bohm scaling).

- The momentum balance equation determining $E_r$ in quasisymmetric helical system with stellarator symmetry is shown to be of $O(\delta^3)$ by using a novel parity operator.