Uniqueness of Laplacian and Brownian motion on Sierpinski carpets

Alexander Teplyaev

University of Connecticut

An Isaac Newton Institute Workshop
Analysis on Graphs and its Applications Follow-up Meeting
26 July to 30 July 2010
The **Annus Mirabilis papers** (from Latin *annus mīrābilis*, "extraordinary year") are the papers of Albert Einstein published in the *Annalen der Physik* scientific journal in 1905. These four articles contributed substantially to the foundation of modern physics and changed views on space, time, and matter. The *Annus Mirabilis* is often called the "Miracle Year" in English or *Wunderjahr* in German. [1]
Einstein’s Annus Mirabilis 1905 papers:

- Matter and energy equivalence ($E = mc^2$)
- Special relativity (Minkowski 1907)
- Photoelectric effect (Nobel prize in Physics 1921)
- Brownian motion
Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n} \mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

(distance $\sim \sqrt{\text{time}}$

"Einstein space–time relation for Brownian motion"
Gaussian transition density:

\[ p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x - y|^2}{4t} \right) \]

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs; Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with \( \text{Ricci} \geq 0 \):

\[ p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp \left( -c \frac{d(x, y)^2}{t} \right) \]

\[ \text{distance} \sim \sqrt{\text{time}} \]
Gaussian:
\[ p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x - y|^2}{4t} \right) \]

Li-Yau Gaussian-type:
\[ p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp \left( -c \frac{d(x, y)^2}{t} \right) \]

Sub-Gaussian:
\[ p_t(x, y) \sim \frac{1}{t_{df/dw}} \exp \left( -c \left( \frac{d(x, y)^{dw}}{t} \right)^\frac{1}{dw-1} \right) \]

\[ distance \sim (time)^{\frac{1}{dw}} \]
Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

Anomalous diffusion: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here $d_w$ is the so-called walk dimension (should be called “walk index” perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.
\[ p_t(x, y) \sim \frac{1}{t^{d_f/d_w}} \exp \left( -c \frac{d(x, y)^{d_w}}{t^{d_w}} \right) \]

\textit{distance} \sim (\textit{time})^{\frac{1}{d_w}}

\( d_f = \) Hausdorff dimension
\( d_w = \) “walk dimension”
\( \frac{2d_f}{d_w} = \) “spectral dimension”

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)
The Sierpinski gasket (left), and a typical nested fractal, the Lindstrøm snowflake (right)
Sierpiński carpet
Wacław Franciszek Sierpiński (Polish pronunciation: [ˈvatɔsfaf fraɲˈt͡ɕt͡ɕɛʃɛk ɕɛrˈpʲiɲskʲi]) (March 14, 1882, Warsaw — October 21, 1969, Warsaw) was a Polish mathematician. He was known for outstanding contributions to set theory (research on the axiom of choice and the continuum hypothesis), number theory, theory of functions and topology. He published over 700 papers and 50 books.

Three well-known fractals are named after him (the Sierpinski triangle, the Sierpinski carpet and the Sierpinski curve), as are Sierpinski numbers and the associated Sierpiński problem.
ASTRONOMIE. — Observations de la comète Mellish, faites à l'Observatoire de Marseille (chercheur de comètes). Note de M. Coggia, présentée par M. B. Baillaud.

<table>
<thead>
<tr>
<th>Dates</th>
<th>Temps moyen de Marseille</th>
<th>Δm</th>
<th>ΔΩ</th>
<th>Nombre de comp.</th>
<th>Δ apparente</th>
<th>Log. fact. parall.</th>
<th>Ω apparente</th>
<th>Log. fact. parall.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Févr. 20</td>
<td>17 h 22 m 14 s</td>
<td>+3° 48' 13&quot;</td>
<td>+6° 36' 9&quot;</td>
<td>15: 5</td>
<td>-1° 30' 8&quot;</td>
<td>+2° 23' 11&quot;</td>
<td>+0° 76' 1&quot;</td>
<td></td>
</tr>
<tr>
<td>≡ 23</td>
<td>17 h 31 m 25 s</td>
<td>-2° 22' 66&quot;</td>
<td>+1° 10' 4&quot;</td>
<td>15: 5</td>
<td>-1° 24' 1&quot;</td>
<td>+2° 10' 24' 2</td>
<td>+0° 76' 2&quot;</td>
<td></td>
</tr>
<tr>
<td>≡ 25</td>
<td>17 h 39 m 58 s</td>
<td>+0° 8' 36&quot;</td>
<td>-8° 4' 6&quot;</td>
<td>15: 5</td>
<td>-1° 17' 7&quot;</td>
<td>+2° 1' 6,6</td>
<td>+0° 76' 3&quot;</td>
<td></td>
</tr>
</tbody>
</table>

Positions moyennes des étoiles de comparaison.

<table>
<thead>
<tr>
<th>Moyenne</th>
<th>Réduction</th>
<th>Moyenne</th>
<th>Réduction</th>
<th>Autorités</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gr.</td>
<td>h m s</td>
<td>à jour</td>
<td>h m s</td>
<td>1915, 0.</td>
</tr>
<tr>
<td>1......</td>
<td>6,3</td>
<td>+0° 64'</td>
<td>17 h 11 m 57 s</td>
<td>-17,1</td>
</tr>
<tr>
<td>2......</td>
<td>8,9</td>
<td>+0° 68'</td>
<td>17 h 21 m 55 s</td>
<td>-17,0</td>
</tr>
<tr>
<td>3......</td>
<td>8,9</td>
<td>+0° 74'</td>
<td>17 h 21 m 55 s</td>
<td>-17,2</td>
</tr>
</tbody>
</table>

La comète est diffuse, irrégulière, sans point brillant ni condensation. Eclat 11e.

ANALYSE MATHÉMATIQUE. — Sur une courbe dont tout point est un point de ramification. Note (1) de M. W. Sierpinski, présentée par M. Émile Picard.

Le but de cette Note est de donner un exemple d'une courbe cantoriennne et jordanienne en même temps, dont tout point est un point de ramification. (Nous appelons point de ramification d'une courbe c un point p de cette courbe, s'il existe trois continus, sous-ensembles de c, ayant deux à deux le point p et seulement ce point commun.)

Soient T un triangle régulier donné ; A, B, C respectivement ses sommets : gauche, supérieur et droit. En joignant les milieux des côtés du triangle T, nous obtenons quatre nouveaux triangles réguliers (fig. 1), dont troi, T1, T2, T3, contenant respectivement les sommets A, B, C sont situés parallèlement à T et le quatrième triangle U contient le centre du triangle T ; nous exclurons tout l'intérieur du triangle U.

Les sommets des triangles T3, T1, T2 nous les désignerons respectivement :

(1) Séance du 1er février 1915.
SÉANCE DU 1er MARS 1915.

les sommets gauches, par $A_0, A_1, A_2$; supérieurs, par $B_0, B_1, B_2$; droits, par $C_0, C_1, C_2$.

Opérons sur chacun de triangles $T_0, T_1, T_2$ comme nous l'avons fait pour le triangle $T$ : nous aurons neuf nouveaux triangles situés parallèlement au triangle $T$, lesquels nous désignerons par

$$T_{\lambda_1\lambda_2}(\lambda_1 = 0, 1, 2; \lambda_2 = 0, 1, 2)$$

et leurs sommets respectivement par $A_{\lambda_1\lambda_2}, B_{\lambda_1\lambda_2}, C_{\lambda_1\lambda_2}$, et trois nouveaux triangles $U_0, U_1, U_2$, situés parallèlement à $U$, dont les intérieurs seront exclus (fig. 2). Avec chacun des triangles $T_{\lambda_1\lambda_2}$ procédons de même et ainsi de suite, en désignant toujours par $A_{\lambda_1\lambda_2...\lambda_n}, B_{\lambda_1\lambda_2...\lambda_n}, C_{\lambda_1\lambda_2...\lambda_n}$ respectivement les sommets de gauche, supérieur et droit du triangle $T_{\lambda_1\lambda_2...\lambda_n}$ et par $T_{\lambda_1...\lambda_{n+1}}, T_{\lambda_2...\lambda_n}$ les nouveaux triangles, contenant respectivement les sommets $A_{\lambda_1...\lambda_n}, B_{\lambda_1...\lambda_n}, C_{\lambda_1...\lambda_n}$; enfin par $U_{\lambda_1...\lambda_n}$ le triangle situé parallèlement à $U$ et inscrit dans $T_{\lambda_1...\lambda_n}$.

Soit $\Theta$ l'ensemble de tous les points du triangle $T$ qui ne sont pas intérieurs à aucun des triangles

$$U, U_0, U_1, U_2, U_{00}, U_{01}, \ldots, U_{\lambda_1\lambda_2...\lambda_n}.$$ 

On voit sans peine que l'ensemble $\Theta$ est un continu non dense dans le plan : c'est donc une courbe cantoriennne.

Soit $\rho$ un point de la courbe $\Theta$ qui n'est pas un sommet du triangle $T$ et d'aucun des triangles $T_{\lambda_1\lambda_2...\lambda_n}$. On voit sans peine qu'il existe une suite infinie d'indices $0, 1, 2$, bien déterminée par le point $\rho$ :

$$\alpha_0, \alpha_1, \alpha_2, \ldots$$

C. R., 1915, 1er Semestre. (T. 160, N° 9.)
de ramification de la courbe $\varnothing$. Quant aux sommets des triangles $T_{\lambda_1, \ldots, \lambda_n}$ (excepté les sommets du triangle $T$), on voit sans peine que dans chacun

![Fig. 3.](image1)

![Fig. 4.](image2)

d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble $\varnothing$.

Donc, tous les points de la courbe $\varnothing$, sauf peut-être les points $A$, $B$, $C$, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

![Fig. 5.](image3)

![Fig. 6.](image4)

points de ramification, il suffit de diviser un hexagone régulier en six triangles réguliers et dans chacun d'eux inscrire une courbe $\varnothing$.

Remarquons qu'on pourrait démontrer sans peine que la courbe $\varnothing$ est une courbe jordaniennne. Or, la courbe $\varnothing$ peut être regardée comme limite d'une suite de lignes brisées dont les premiers termes représentent les figures 3, 4, 5 et 6.
de ramification de la courbe $\omega$. Quant aux sommets des triangles $T_1, \ldots, T_n$ (excepté les sommets du triangle $T$), on voit sans peine que dans chacun

![Fig. 3](image3.png) ![Fig. 4](image4.png)

d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble $\omega$.

Donc, tous les points de la courbe $\omega$, sauf peut-être les points $A$, $B$, $C$, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

![Fig. 5](image5.png) ![Fig. 6](image6.png)

points de ramification, il suffit de diviser un hexagone régulier en six triangles réguliers et dans chacun d'eux inscrire une courbe $\omega$.

Remarquons qu'on pourrait démontrer sans peine que la courbe $\omega$ est une courbe jordaniennne. Or, la courbe $\omega$ peut être regardée comme limite d'une suite de lignes brisées dont les premiers termes représentent les figures 3, 4, 5 et 6.
Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. To cite this article: R. Grigorchuk, Z. Šunič, C. R. Acad. Sci. Paris, Ser. I 344 (2006).

Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L’automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3
Asymptotic aspects of Schreier graphs and Hanoi Towers groups

Rostislav Grigorchuk\(^1\), Zoran Šunič

Department of Mathematics, Texas A&M University, MS-3368, College Station, TX, 77843-3368, USA

Received 23 January, 2006; accepted after revision ++++

Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior.


Figure 1. The automaton generating $H(4)$ and the Schreier graph of $H(3)$ at level 3 / L’automate engendrant $H(4)$ et le graphe de Schreier de $H(3)$ au niveau 3
Noam Chomsky

From Wikipedia, the free encyclopedia

"Chomsky" redirects here. For other persons of the same name, see Chomsky (surname).

Avram Noam Chomsky
(pronounced /ˈnoʊm ˈtʃɒmski/; born December 7, 1928) is an American linguist, philosopher,[2][3][4][5] cognitive scientist, political activist, author, and lecturer. He is an Institute Professor and professor emeritus of linguistics at the Massachusetts Institute of Technology.[6] Chomsky is well
Main classes of fractals considered

- [0, 1]
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- Dirichlet metric measure spaces with heat kernel estimates (DMMS+HKE)
Figure: Sierpiński gasket and Lindstrøm snowfalke (nested fractals), p.c.f., finitely ramified
Figure: Diamond fractals, non-p.c.f., but finitely ramified
Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified
Figure: Sierpiński carpet, infinitely ramified
Initial motivation

Main early results


Summary: We investigate the asymptotic motion of a random walker, which at time $n$ is at $X(n)$, on certain ‘fractal lattices’. For the ‘Sierpiński lattice’ in dimension $d$ we show that, as $l \to \infty$, the process $Y_l(t) \equiv X([((d + 3)^l)t])/2^l$ converges in distribution (so that, in particular, $|X(n)| \sim n^\gamma$, where $\gamma = (\ln 2)/\ln(d + 3)$) to a diffusion on the Sierpin’ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple ‘renormalization group’ type argument, involving self-similarity and ‘decimation invariance’.


S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* (1989)


The classical diffusion process was first studied by Einstein, and later a mathematical theory was developed by Wiener, Kolmogorov, Levy et al. One of the basic principle is that displacement in a small time is proportional to the square root of time. This law is related to the properties of the Gaussian transition density and the heat equation.

On fractals diffusions have to obey scaling laws what are different from the classical Gaussian diffusion, but are of so called sub-Gaussian type. In some situations the diffusion, and therefore the correspondent Laplace operator, is uniquely determined by the geometry of the space. (Recently it was proved for Sierpinski square and its generalizations, a joint work with M. T. Barlow, R. F. Bass, T. Kumagai).

As a consequence, there are uniquely defined spectral and walk dimensions, which are related by so called Einstein relation and determine the behavior of the natural diffusion processes by (these dimensions are different from the well known Hausdorff dimension, which describes the distribution of the mass in a fractal).

\[ 2d_f / d_s = d_f + \tilde{\zeta} = d_w \]
Abstract. We prove that, up to scalar multiples, there exists only one local regular Dirichlet form on a generalized Sierpiński carpet that is invariant with respect to the local symmetries of the carpet. Consequently, for each such fractal the law of Brownian motion is uniquely determined and the Laplacian is well defined.

Keywords. Sierpiński carpet, fractals, diffusions, Brownian motion, uniqueness, Dirichlet forms

1. Introduction

The standard Sierpiński carpet $F_{SC}$ is the fractal that is formed by taking the unit square, dividing it into 9 equal subsquares, removing the central square, dividing each of the 8 remaining subsquares into 9 equal smaller pieces, and continuing. In [3] two of the authors of this paper gave a construction of a Brownian motion on $F_{SC}$. This is a diffusion (that is, a continuous strong Markov process) which takes its values in $F_{SC}$, and which is non-degenerate and invariant under all the local isometries of $F_{SC}$.

Subsequently, Kusuoka and Zhou in [27] gave a different construction of a diffusion on $F_{SC}$, which yielded a process that, as well as having the invariance properties of the Brownian motion constructed in [3], was also scale invariant. The proofs in [3, 27] also work for fractals that are formed in a similar manner to the standard Sierpiński carpet: we call these generalized Sierpiński carpets (GSCs). In [5] the results of [3] were extended to GSCs embedded in $\mathbb{R}^d$ for $d \geq 3$. While [3, 5] and [27] both obtained their diffusions as limits of approximating processes, the type of approximation was different: [3, 5] used a sequence of time changed reflecting Brownian motions, while [27] used a sequence of Markov chains.

Mathematics Subject Classification (2010): Primary 60G18; Secondary 60J35, 60J60, 28A80
1. Introduction

Let $F$ be a GSC and $\mu$ the usual Hausdorff measure on $F$. Let $\mathcal{E}$ be the set of non-zero local regular conservative Dirichlet forms on $L^2(F, \mu)$ which are invariant with respect to all the local symmetries of $F$.

**Theorem 1.1.** Let $F \subset \mathbb{R}^d$ be a GSC. Then, up to scalar multiples, $\mathcal{E}$ consists of at most one element. Further, this one element of $\mathcal{E}$ is self-similar.

**Proposition 1.2.** The Dirichlet forms constructed in [BB89, BB99] and [KZ92] are in $\mathcal{E}$ (and the approximations converge).

**Corollary 1.3.** The Dirichlet forms constructed in [BB89, BB99] and [KZ92] are (up to a constant) the same, and satisfy scale invariance (i.e. self-similar).

**Corollary 1.4.** If $X$ is a continuous non-degenerate symmetric strong Markov process, whose state space is $F$, and whose Dirichlet form is invariant with respect to the local symmetries of $F$, then the law of $X$, which is a Feller process, is uniquely defined, up to scalar multiples of the time parameter, for each initial point $x \in F$. 


We do not assume the heat kernel exists, or even that the semi-group is Feller, or that the Dirichlet form is irreducible.

The idea of our proof is the following. The main work is showing that if $\mathcal{A}, \mathcal{B}$ are any two Dirichlet forms in $\mathcal{E}$, then they are comparable. We then let $\lambda$ be the largest positive real such that $\mathcal{C} = \mathcal{A} - \lambda \mathcal{B} \geq 0$. If $\mathcal{C}$ were also in $\mathcal{E}$, then $\mathcal{C}$ would be comparable to $\mathcal{B}$, and so there would exist $\varepsilon > 0$ such that $\mathcal{C} - \varepsilon \mathcal{B} \geq 0$, contradicting the definition of $\lambda$. In fact we cannot be sure that $\mathcal{C}$ is closed, so instead we consider $\mathcal{C}_\delta = (1 + \delta) \mathcal{A} - \lambda \mathcal{B}$, which is easily seen to be in $\mathcal{E}$. We then need uniform estimates in $\delta$ to obtain a contradiction.

A key point here is that the constants in the Harnack inequality, and consequently also the heat kernel bounds, only depend on the GSC $F$, and not on the particular element of $\mathcal{E}$. This means that we need to be careful about the dependencies of the constants.
Our general (distant) aim is to use this result:

Theorem (Grigor’yan and Telcs). Let \((X, d, \mu, \mathcal{E}, \mathcal{F})\) be a MMD space. (Note that the assumption includes the facts that \(d\) is geodesic and \((\mathcal{E}, \mathcal{F})\) is conservative.) Then TFAE, and the constants in each implication are effective:

(a) \(X\) satisfies \((VD), (EHI)\) and \((RES(H))\).
(b) \(X\) satisfies \((VD), (EHI)\) and \((E(H))\).
(c) \(X\) satisfies \((HK(H, \beta_1, \beta_2, c_0))\).

The equivalence of the “global” version (i.e. each condition holds for \(t \in (0, \infty), R \geq 0\)) also holds.

Here \(H : [0, 2] \to [0, \infty)\) is a strictly increasing function which (for reasons which will be apparent later) is called the time scaling function. We introduce the following assume that there exist \(C_2, \ldots, C_5 > 0\), and \(\beta_1 > 1\) such that

\[
(H(1) \in [C_2, C_3], \text{ and})
\]

\[
[H(2R) \leq C_4 H(R) \text{ for all } 0 < R \leq 1.]
\]

\[
[H(R) / H(r) \geq C_5 (R/r)^{\beta_1} \text{ for all } 0 < r < R \leq 2.]
\]

Here (TD) refers to ‘time doubling’ and (FTG) to ‘fast time growth’.
2. Preliminaries

2.1. Some general properties of Dirichlet forms.

Theorem 2.1. Let \((A, \mathcal{F}), (B, \mathcal{F})\) be regular local conservative irreducible Dirichlet forms on \(L^2(F, m)\) and

\[ A(u, u) \leq B(u, u) \quad \text{for all } u \in \mathcal{F}. \]

Let \(\delta > 0\), and \(E = (1 + \delta)B - A\). Then \((E, \mathcal{F})\) is a regular local conservative irreducible Dirichlet form on \(L^2(F, m)\).

Since \(E\) is local regular, \(E(f, f)\) can be written in terms of a measure \(\Gamma(f, f)\), the energy measure of \(f\), as follows. Let \(\mathcal{F}_b\) be the elements of \(\mathcal{F}\) that are essentially bounded. If \(f \in \mathcal{F}_b\), \(\Gamma(f, f)\) is the unique smooth Borel measure on \(F\) satisfying

\[ \int_{\mathcal{F}} gd\Gamma(f, f) = 2E(f, fg) - E(f^2, g), \quad g \in \mathcal{F}_b. \]

Lemma 2.2. If \(E\) is a local regular Dirichlet form with domain \(\mathcal{F}\), then for any \(f \in \mathcal{F} \cap L^\infty(F)\) we have \(\Gamma(f, f)(A) = 0\), if \(A = \{x \in F : f(x) = 0\}\).
We call a function \( u : \mathbb{R}_+ \times F \to \mathbb{R} \) caloric in \( D \) in probabilistic sense if
\[
u(t, x) = \mathbb{E}^x[f(X_{t\wedge \tau_D})]
\]
for some bounded Borel \( f : F \to \mathbb{R} \), which is the solution to the heat equation with boundary data defined by \( f(x) \) outside of \( D \) and the initial data defined by \( f(x) \) inside of \( D \). Let \( T_t \) be the semigroup of \( X \) killed on exiting \( D \), which can be either defined probabilistically as above or, equivalently, in the Dirichlet form sense according to Theorems 4.4.3 and A.2.10 in [FOT].

**Proposition 2.3.** Let \((\mathcal{E}, \mathcal{F})\) and \( D \) satisfy the above conditions, and let \( f \in \mathcal{F} \) be bounded and \( t \geq 0 \). Then
\[
\mathbb{E}^x[f(X_{t\wedge \tau_D})] = h(x) + T_t f_D
\]
q.e., where \( h(x) = \mathbb{E}^x[f(X_{\tau_D})] \) is the harmonic function that consides with \( f \) on \( D^c \), and \( f_D(x) = f(x) - h(x) \).
2.2. Generalized Sierpinski carpets. Let $d \geq 2$, $F_0 = [0,1]^d$, and let $L_F \in \mathbb{N}$, $L_F \geq 3$, be fixed. For $n \in \mathbb{Z}$ let $Q_n$ be the collection of closed cubes of side $L_F^{-n}$ with vertices in $L_F^{-n}\mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$, set $Q_n(A) = \{Q \in Q_n : \text{int}(Q) \cap A \neq \emptyset\}$. Let $\Psi_Q$ be the orientation preserving affine map from $F_0$ onto $Q$. Let $1 \leq m_F \leq L_F^d$ be an integer, and let $F_1$ be the union of $m_F$ distinct elements of $Q_1(F_0)$.

- (H1) (Symmetry) $F_1$ is preserved by the isometries of the unit cube $F_0$.
- (H2) (Connectedness) $\text{Int}(F_1)$ is connected.
- (H3) (Non-diagonality) Let $m \geq 1$ and $B \subset F_0$ be a cube of side length $2L_F^{-m}$, which is the union of $2^d$ distinct elements of $Q_m$. Then if $\text{int}(F_1 \cap B)$ is non-empty, it is connected.
- (H4) (Borders included) $F_1$ contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = ... = x_d = 0\}$.

Given $S \in \mathcal{S}_n$, there is the folding map $\varphi_S : F \to S$.

For $f : S \to \mathbb{R}$ and $g : F \to \mathbb{R}$ we define the unfolding and restriction operators by $U_S f = f \circ \varphi_S$, $R_S g = g|_S$. 
Definition 2.4. We define the length and mass scale factors of $F$ to be $L_F$ and $m_F$ respectively. The Hausdorff dimension of $F$ is $d_f = d_f(F) = \log m_F / \log L_F$.

Let $D_n$ be the network of diagonal crosswires obtained by joining each vertex of a cube $Q \in \mathcal{Q}_n$ to a vertex at the center of the cube by a wire of unit resistance. Write $R_n^D$ for the resistance across two opposite faces of $D_n$. There exists $\rho_F$ and $C_i$, depending only on the dimension $d$, such that $\rho_F \leq L_F^2 / m_F$ and 

$$C_1 \rho_F^n \leq R_n^D \leq C_2 \rho_F^n.$$
2.3. $F$-invariant Dirichlet forms. Let $(\mathcal{E}, \mathcal{F})$ be a regular local Dirichlet form on $L^2(F, \mu)$. Let $S \in \mathcal{S}_n$. We set

$$\mathcal{E}^S(g, g) = \frac{1}{m_F^n} \mathcal{E}(U_S g, U_S g).$$

and define the domain of $\mathcal{E}^S$ to be $\mathcal{F}^S = \{ g : g \text{ maps } S \text{ to } \mathbb{R}, U_S g \in \mathcal{F} \}$.

**Definition 2.5.** Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(F, \mu)$. We say that $\mathcal{E}$ is invariant with respect to all the local symmetries of $F$ ($F$-invariant or $\mathcal{E} \in \mathcal{E}$) if

- (1) If $S \in \mathcal{S}_n(F)$, then $U_S R_S f \in \mathcal{F}$ for any $f \in \mathcal{F}$.
- (2) Let $n \geq 0$ and $S_1, S_2$ be any two elements of $\mathcal{S}_n$, and let $\Phi$ be any isometry of $\mathbb{R}^d$ which maps $S_1$ onto $S_2$. If $f \in \mathcal{F}^{S_2}$, then $f \circ \Phi \in \mathcal{F}^{S_1}$ and $\mathcal{E}^{S_1}(f \circ \Phi, f \circ \Phi) = \mathcal{E}^{S_2}(f, f)$.
- (3) $\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$ for all $f \in \mathcal{F}$

**Lemma 2.6.** Let $(\mathcal{A}, \mathcal{F}_1), (\mathcal{B}, \mathcal{F}_2) \in \mathcal{E}$ with $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{A} \geq \mathcal{B}$. Then $\mathcal{C} = (1 + \delta)\mathcal{A} - \mathcal{B} \in \mathcal{E}$ for any $\delta > 0$. 

\[ \Theta f = \frac{1}{m_F^n} \sum_{S \in \mathcal{S}_n(F)} U_S R_S f. \]

Note that \( \Theta \) is a projection operator because \( \Theta^2 = \Theta \). It is bounded on \( C(F) \) and is an orthogonal projection on \( L^2(F, \mu) \).

**Proposition 2.7.** Assume that \( \mathcal{E} \) is a local regular Dirichlet form on \( F \), \( T_t \) is its semigroup, and \( U_S R_S f \in \mathcal{F} \) whenever \( S \in \mathcal{S}_n(F) \) and \( f \in \mathcal{F} \). Then the following, for all \( f, g \in \mathcal{F} \), are equivalent:

(a): \( \mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f) \)

(b): \( \mathcal{E}(\Theta f, g) = \mathcal{E}(f, \Theta g) \)

(c): \( T_t \Theta f = \Theta T_t f \)
3. The Barlow-Bass and Kusuoka-Zhou Dirichlet forms

Theorem 3.1. Each $\mathcal{E}_{BB}$ and $\mathcal{E}_{KZ}$ is in $\mathcal{C}$.

4. Diffusions associated with $F$-invariant Dirichlet forms

Let $X = X^{(\mathcal{E})}$ be an $\mathcal{E}$-diffusion, $T_t = T_t^{(\mathcal{E})}$ be the semigroup of $X$ and $\mathbb{P}^x = \mathbb{P}^{x,(\mathcal{E})}$, $x \in F - \mathcal{N}_0$, the associated probability laws. Here $\mathcal{N}_0$ is a properly exceptional set for $X$. Ultimately we will be able to define $\mathbb{P}^x$ for all $x \in F$, so that $\mathcal{N}_0 = \emptyset$.

4.1. Reflected processes and Markov property.

Theorem 4.1. Let $S \in \mathcal{S}_n(F)$. Let $Z = \varphi_S(X)$. Then $Z$ is a $\mu_S$-symmetric Markov process with Dirichlet form $(\mathcal{E}^S, \mathcal{F}^S)$, and semigroup $T_t^Z f = R_S T_t U_S f$. Write $\mathbb{P}^y$ for the laws of $Z$; these are defined for $y \in S - \mathcal{N}_2^Z$, where $\mathcal{N}_2^Z$ is a properly exceptional set for $Z$. There exists a properly exceptional set $\mathcal{N}_2$ for $X$ such that for any Borel set $A \subset F$,

$$\mathbb{P}^{\varphi_S(x)}(Z_t \in A) = \mathbb{P}^x(X_t \in \varphi^{-1}_S(A)), \quad x \in F - \mathcal{N}_2.$$
The half-face $A_1$ corresponds to a “slide move”, and the half-face $A'_1$ corresponds to a “corner move”, analogues of the “corner” and “knight’s” moves in [BB89].
A half-face knight's move.
4.2. Moves by $Z$ and $X$. The key idea, as in [BB99], is to prove that certain 'moves' of the process in $F$ have probabilities which can be bounded below by constants depending only on the dimension $d$. We begin by looking at the process $Z = \varphi_S(X)$ for some $S \in S_n$, where $n \geq 0$.

Let $1 \leq i, j \leq d$, with $i \neq j$, assume $n = 0$ and $S = F$, and

$H_i(t) = \{x = (x_1, \ldots, x_d) : x_i = t\}$, $t \in \mathbb{R}$;

$L_i = H_i(0) \cap [0, 1/2]^d$;

$M_{ij} = \{x \in [0, 1]^d : x_i = 0, \frac{1}{2} \leq x_j \leq 1, \text{ and } 0 \leq x_k \leq \frac{1}{2} \text{ for } k \neq j\}$.

$$\partial_e S = S \cap (\cup_{i=1}^d H_i(1)), \quad D = S - \partial_e S.$$ 

**Proposition 4.2.** There exists a constant $q_0$, depending only on the dimension $d$, such that for any $n \geq 0$

$$\bar{\mathbb{P}}^x(T_{L_i}^Z < \tau_D^Z) \geq q_0, \quad x \in L_i \cap E_D,$$

$$\bar{\mathbb{P}}^x(T_{M_{ij}}^Z < \tau_D^Z) \geq q_0, \quad x \in L_i \cap E_D.$$
$D(y)$
4.5. **Elliptic Harnack inequality. (Definition)**

$X$ satisfies the *elliptic Harnack inequality* if there exists a constant $c_1$ such that the following holds: for any ball $B(x, R)$, whenever $u$ is a non-negative harmonic function on $B(x, R)$ then there is a quasi-continuous modification $\tilde{u}$ of $u$ that satisfies

$$\sup_{B(x, R/2)} \tilde{u} \leq c_1 \inf_{B(x, R/2)} \tilde{u}.$$
Lemma 4.10. Let $r \in (0, 1)$, and $h$ be bounded and harmonic in $B = B(x_0, r)$. Then there exists $\theta > 0$ such that

$$|h(x) - h(y)| \leq C\left(\frac{|x - y|}{r}\right)^\theta \left(\sup_B |h|\right), \quad x, y \in B(x_0, r/2), \quad x \sim y.$$

Proposition 4.11. There exists a set $\mathcal{N}$ of $\mathcal{E}$-capacity 0 such that the Lemma above holds for all $x, y \in B(x_0, r/2) - \mathcal{N}$.

Proposition 4.12. EHI holds for $\mathcal{E}$, with constants depending only on $F$.

Corollary 4.13. If $\mathcal{E} \in \mathcal{E}$ then

(a) $\mathcal{E}$ is irreducible;
(b) if $\mathcal{E}(f, f) = 0$ then $f$ is a.e. constant;
(c) $||\mathcal{E}|| > 0$, where $||\mathcal{E}||$ is the effective resistance between two opposite faces of the GSC.
4.6. Resistance estimates. Let now $E \in \mathcal{E}_1$. Let $S \in \mathcal{S}_n$ and let $\gamma_n = \gamma_n(E)$ be the conductance across $S$. That is, if $S = Q \cap F$ for $Q \in \mathcal{Q}_n(F)$ and $Q = \{a_i \leq x_i \leq b_i, i = 1, \ldots, d\}$, then

$$\gamma_n = \inf \{E^S(u, u) : u \in \mathcal{F}^S, u \mid_{\{x_1 = a_1\}} = 0, u \mid_{\{x_1 = b_1\}} = 1\}.$$ 

Note that $\gamma_n$ does not depend on $S$, and that $\gamma_0 = 1$. Write $v_n = v_n^E$ for the minimizing function. We remark that from the results in [BB3, McG] we have

$$C_1 \rho_n^F \leq \gamma_n(E_{BB}) \leq C_2 \rho_n^F.$$ 

Proposition 4.14. Let $E \in \mathcal{E}_1$. Then for $n, m \geq 0$

$$\gamma_n + m(E) \geq C_1 \gamma_m(E) \rho_n^F.$$ 

We define a ‘time scale function’ $H$ for $E$...

We say $E$ satisfies the condition RES$(H, c_1, c_2)$ if for all $x, r \in (0, L_F^{-1})$, 

$$c_1 \frac{H(r)}{r^\alpha} \leq R_{\text{eff}}(B(x_0, r), B(x_0, 2r)^c) \leq c_2 \frac{H(r)}{r^\alpha}. \quad [\text{RES}(H, c_1, c_2)]$$

Proposition 4.15. There exist constants $C_1, C_2$, depending only on $F$, such that $E$ satisfies RES$(H, C_1, C_2)$. 

4.7. Exit times, heat kernel and energy estimates. We write $h$ for the inverse of $H$, and $V(x, r) = \mu(B(x, r))$. We say that $p_t(x, y)$ satisfies $HK(H; \eta_1, \eta_2, c_0)$ if for $x, y \in F$, $0 < t \leq 1$,

\[ p_t(x, y) \geq c_0^{-1}V(x, h(t))^{-1}\exp(-c_0(H(d(x, y))/t)^{\eta_1}), \]
\[ p_t(x, y) \leq c_0V(x, h(t))^{-1}\exp(-c_0^{-1}(H(d(x, y))/t)^{\eta_2}). \]

Theorem 4.16 (GT (also BBKT-supplement)).

Let $H : [0, 2] \to [0, \infty)$ be a strictly increasing function that satisfies ...

Then TFAE:

(a) $(\mathcal{E}, \mathcal{F})$ satisfies $(VD)$, $(EHI)$ and $(RES(H, c_1, c_2))$

(b) $(\mathcal{E}, \mathcal{F})$ satisfies $(HK(\alpha, H; \eta_1, \eta_2, c_0))$

Further the constants in each implication are effective.

By saying that the constants are 'effective' we mean that if, for example (a) holds, then the constants $\eta_i, c_0$ in (b) depend only on the constants $c_i$ in (a), and the constants in (VD), (EHI) and ...
Let
\[ J_r(f) = r^{-\alpha} \int_F \int_{B(x,r)} |f(x) - f(y)|^2 d\mu(x) d\mu(y), \]
\[ N_r^H(f) = H(r)^{-1} J_r(f), \]
\[ N_H(f) = \sup_{0 < r \leq 1} N_r^H(f), \]
\[ W_H = \{ u \in L^2(F, \mu) : N_H(f) < \infty \}. \]

Theorem 4.18 (KS,BBKT). Let \( H \) satisfy ... Suppose \( p_t \) satisfies \( HK(H; \eta_1, \eta_2, C_0) \). Then
\[ C_1 \mathcal{E}(f, f) \leq \limsup_{j \to \infty} N_r^H(f) \leq N_H(f) \leq C_2 \mathcal{E}(f, f) \quad \text{for all } f \in W_H, \]
where the constants \( C_i \) depend only on the constants in \( HK(H; \eta_1, \eta_2, C) \), and in ... Further,
\[ \mathcal{F} = W_H. \]
5. Uniqueness

Definition 5.1. Let $A, B \in \mathcal{E}$. We say $A \leq B$ if

$$B(u, u) - A(u, u) \geq 0 \text{ for all } u \in W.$$ 

For $A, B \in \mathcal{E}$ define

$$\sup(B|A) = \sup \left\{ \frac{B(f, f)}{A(f, f)} : f \in W \right\},$$

$$h(A, B) = \log \left( \frac{\sup(B|A)}{\sup(A|B)} \right).$$

Note that $h$ is Hilbert’s projective metric and we have $h(\theta A, B) = h(A, B)$ for any $\theta \in (0, \infty)$, and $h(A, B) = 0$ if and only if $A$ is a nonzero constant multiple of $B$.

Theorem 5.2. There exists a constant $C_F$, depending only on the GSC $F$, such that if $A, B \in \mathcal{E}$ then

$$h(A, B) \leq C_F.$$
Appendix: some spectral results on finitely ramified fractals with symmetries

Physical consequences of complex dimensions of fractals

E. Akkermans\textsuperscript{1(a)}, G. V. Dunne\textsuperscript{2(b)} and A. Teplyaev\textsuperscript{3}

\textsuperscript{1} Department of Applied Physics and Physics, Yale University - New Haven, CT 06520, USA
\textsuperscript{2} Department of Physics, University of Connecticut - Storrs, CT 06269, USA
\textsuperscript{3} Department of Mathematics, University of Connecticut - Storrs, CT 06269, USA

received 5 August 2009; accepted in final form 2 November 2009
published online 3 December 2009
Our main result is the identification and characterization of a new oscillating behavior of $Z(t)$ at small $t$, which has implications for various physical quantities. Such oscillations do not exist for smooth manifolds, or even for quantum graphs. We apply these considerations to the concrete case of quantum mesoscopic systems [6], and show that the oscillating behavior can be directly observed in spectral quantities such as the fluctuations of the number of energy levels and the Wigner time delay. We also relate the electric conductance $g$, the associated weak localization corrections $\Delta g$, and universal conductance fluctuations $\delta g^2$ to the fractal zeta function.

We first recall some basic definitions and facts about deterministic fractals. As opposed to Euclidean spaces characterized by translation symmetry, self-similar (fractal) structures possess a dilatation symmetry of their physical properties, each characterized by a specific fractal dimension. To illustrate them, we consider throughout this letter the family of diamond fractals (see fig. 1), but keeping in mind that our results apply to a much broader class of fractals, including the Sierpinski gasket. At each step $n$ of the iteration, we characterize a fractal by its total length $L_n$, the number of sites $N_n$, and the diffusion time $T_n$. Scaling of these dimensionless quantities allows to define the corresponding Hausdorff $d_h$, spectral $d_s$, and walk $d_w$ dimensions according to

$$d_h = \frac{\ln N_n}{\ln L_n}, \quad d_w = \frac{\ln T_n}{\ln L_n}, \quad d_s = 2 \frac{\ln N_n}{\ln T_n},$$

(1)

where the limit $n \to \infty$ is understood. These three dimensions are thus related by $d_s = 2 d_h / d_w$.

To obtain the heat kernel of a fractal, let us recall the corresponding expression for an Euclidean system of space dimension $d$. We consider the diffusion equation $-\Delta \psi_k(r) = \epsilon_k \psi_k(r)$, where the diffusion coefficient is set to unity, without yet specifying boundary conditions. The probability $P(r, r', t)$ to diffuse, in time $t$, from an initial point $r$ to a final point $r'$, is given by the Green’s function defined in an arbitrary volume $\Omega$:

$$P(r, r', t) = \theta(t) \sum_k \psi_k^* (r) \psi_{k,j} (r') e^{-E_k t}.$$  

Here $g_k$ is a degeneracy factor generally different from unity (e.g. on a sphere [22]), except for one dim. diffusion on a finite interval. The heat kernel $Z(t)$ is defined for $t > 0$:

$$Z(t) = \int_{\Omega} P(r, r, t) \, dr = \sum_k g_k e^{-E_k t}.$$  

(2)

The spectral zeta function is defined by a Mellin-Laplace transform of the heat kernel

$$\zeta(s, \gamma) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s Z(t) e^{-\gamma t} = \sum_k \frac{g_k}{(E_k + \gamma)^s},$$

(3)

Many quantities are derived directly from the spectral zeta function. E.g., the spectral determinant $S(\gamma)$ is [7]

$$S(\gamma) = \det(-\Delta + \gamma) = \exp \left[-\frac{d}{ds} \zeta(s, \gamma) \big|_{s=0} \right],$$

(4)

which follows directly from the analytic continuation of $\zeta(s, \gamma)$ in the complex $s$ plane as a meromorphic function analytic at $s = 0$, and the identity $\frac{d}{ds} \lambda^{-s} \big|_{s=0} = - \ln \lambda$. For example, from the spectral determinant, we deduce the density of states: $\rho(E) = -\frac{1}{\pi} \lim_{\lambda \to \infty} \text{Im} \frac{d}{ds} \ln S(\gamma)$, with $\gamma = -E + i\epsilon$. This can also be written [23] in terms of the on-shell $S$-matrix $S(E)$ by the Birman-Krein formula $\rho(E) = \frac{1}{4 \pi \hbar^2} \ln \det S(-E)$, also defining the Wigner time delay: $\tau(E) = -i \hbar \frac{d}{dE} \ln \det S(-E)$.

To generalize (2) to a fractal, we consider the probability $P(r, t)$ to diffuse over a distance $r$ in a time $t$ (with obvious notations). Scaling properties of diffusion are expressed using the definition (1) of the walk dimension $d_w$ through the scaling transformation, $P(\lambda r, \lambda^{d_w} t) = P(r, t)$, for any scaling factor $\lambda$ of the length, so that the probability is of the form $P(r, t) = f(\lambda^{d_w} / t)$, where $f$ is some unknown function. In addition, the normalization condition, $\int d^{d_w} r P(r, t) = 1$, and the change $u = r / t^{1/d_w}$, lead to the general scaling form

$$P(r, t) = \frac{1}{t^{d_w / d_h}} f(\lambda^{d_w} / t).$$

(5)

This implies that diffusion on a fractal is anomalous in the sense that the usual Euclidean relation $\langle r^2 (t) \rangle \propto t$ for long enough times, is now replaced by $\langle r^2 (t) \rangle \propto t^{2/d_w}$: hence the name “anomalous random walk dimension” for $d_w$. Then, relations (1) imply the well-known result $P(0, t) \propto t^{-d_w / 2}$ for the leading term of the return probability which is driven by the spectral dimension $d_h$, rather than by the Hausdorff dimension $d_h$. Generalizing (2), the heat kernel of a diamond fractal can be obtained by noticing that the spectrum of diamond fractals is the union of two sets of eigenvalues. One set is composed of the non degenerate eigenvalues $\pi^2 k^2$, for $k = 1, 2, \ldots$. This corresponds to the spectrum of the diffusion equation defined on a finite one-dimensional interval of unit length, with Dirichlet
boundary conditions. The second ensemble contains iterated eigenvalues, \( \pi^2 k^2 L_n^{d_w} \), obtained by rescaling dimensionless length \( L_n \) and time \( T_n \) at each iteration \( n \) according to \( L_n^{d_w} = T_n \), given in (1). To proceed further, we use the explicit scaling of the length \( L_n = t_n \) upon iteration (see table 1). These iterated eigenvalues have an exponentially large degeneracy, given at each step, by \( B L_n^{d_h} = B (t_n^{d_h}) \), where \( B = (t_n^{-d_h} - 1) \) is the branching factor of the fractal (see fig. 1), and the integer \( t_n \) is the number of links into which a given link is divided.

The exponential growth of the degeneracy plays a crucial role in our analysis. By contrast, on an \( N \)-dimensional sphere the degeneracy grows as a polynomial, of order \( N - 1 \) [22]. Finally, the diamond heat kernel \( Z_D(t) \) is the sum of contributions of the two sets of eigenvalues:

\[
Z_D(t) = \sum_{k=1}^{\infty} e^{-k^2 \pi^2 t} + B \sum_{n=0}^{\infty} L_n^{d_h} \sum_{k=1}^{\infty} e^{-k^2 \pi^2 L_n^{d_w}}. \tag{6}
\]

The associated zeta function \( \zeta_D(s) \), from (3) at \( \gamma = 0 \), is

\[
\zeta_D(s) = \frac{\zeta_R(2s)}{\pi^{2s}} \left( 1 + B \sum_{n=0}^{\infty} L_n^{d_h-d_w} s_n \right) = \frac{\zeta_R(2s)}{\pi^{2s}} p_n^{1-d_w} \left( \frac{1}{1-t_n^{d_h-d_w}} \right), \tag{7}
\]

where \( \zeta_R(2s) \) is the Riemann zeta function. Note that a very similar structure arises for the Sierpinski gasket [14], with the Riemann zeta function factor replaced by another zeta function. \( \zeta_D(s) \) has complex poles given by

\[
s_m = \frac{d_h}{d_w} + \frac{2\pi m}{d_w} \text{ln} l = \frac{d_h}{2} + \frac{2\pi m}{d_w} \text{ln} l, \tag{8}
\]

where \( m \) is an integer. The origin of these complex poles is clearly the exponential degeneracy factors. The complex poles have been identified with complex dimensions for fractals [13,14].

By an inverse Mellin transform, we can write the heat kernel as \( Z_D(t) = \Gamma(s_m) \zeta_R(2s_m)/\pi^{2s_m} \). Then the leading small time behavior comes from the pole of \( \zeta_D(s) \) at \( s = s_0 = d_w/2 \), allowing the anticipated time decreasing function \( \sim t^{-d_w/2} \). The pole of \( \zeta_D(s) \) at \( s = 1/2 \) (coming from the \( \zeta_R(2s) \) factor) has zero residue for all diamonds, and so does not contribute to the short time behavior of \( Z_D(t) \). (Remarkably, this vanishing of the residue at \( s = 1/d_w \) also applies to the analogous zeta function on the Sierpinski gasket [14].) The pole of \( \Gamma(s) \) at \( s = 0 \) gives a constant contribution, \( \zeta_D(0) \), to \( Z_D(t) \). But the really surprising new behavior comes from the complex poles in (8), leading to the oscillatory behavior:

\[
Z_D(t) \sim \frac{(d_h - 1)}{t^{d_h/2}} \left( a_0 + 2Re \left( a_1 t^{-2\pi/(d_w \text{ln} l)} \right) \right) + \zeta_D(0) + \ldots, \tag{9}
\]

where we have defined \( a_m = \Gamma(s_m)\zeta_R(2s_m)/\pi^{2s_m} \). The leading term \( \sim t^{-d_w/2} \) is therefore multiplied by a periodic function of the form \( a_1 \cos(\text{ln} t^{s_1}) \) for \( s_1 \), \( \sin(\text{ln} t^{s_1}) \), where \( a_{1,r,i} \) are respectively the real and imaginary parts of \( a_1 \), and \( s_1 = 2\pi/\text{ln} t^{d_w} \). The oscillations of \( Z_D(t) \) are represented in fig. 2, and we note that the higher complex poles give much smaller contributions. Similar behavior has been found numerically for the Sierpinski gasket [24]; from our work, we further find explicit expressions for the coefficients, also in the Sierpinski case.

In principle, all spectral properties can be derived from the heat kernel (6), or from the associated zeta function \( \zeta_D(s) \) in (7), even though those are not directly accessible physical quantities. For example, the constant term \( \zeta_D(0) \) in (9) leads to a topological term \( \zeta_D(0)\delta(E) \) in the density of states. More interestingly, the oscillations of \( Z_D(t) \) lead to oscillatory behavior in physical quantities.

We give an explicit example of one such quantity, in quantum mesoscopic systems. The fluctuation \( \Sigma^2(E) \) of the number of levels within an energy interval of width \( E \) is defined by the variance, \( \Sigma^2(E) = \overline{N^2(E) - N(E)^2} \), of the integrated density of states (the counting function). In the diffusion approximation, one can express \( \Sigma^2(E) \) directly in terms of the heat kernel through [6]

\[
\Sigma^2(E) = 4 \pi^2 \int_0^{\infty} \frac{dt}{t} Z_D(t) \sin^2 \left( \frac{Et}{2} \right). \tag{10}
\]
SPECTRAL ANALYSIS ON INFINITE SIERPINSKI FRACTAFOLDS

ROBERT S. STRICHARTZ AND ALEXANDER TEPLYAEV

Date: July 27, 2010. Research supported in part by the National Science Foundation, grants DMS-0652440 (first author) and DMS-0505622 (second author).
Our aim is a “Plancherel formula”:
\[
f(x) = \int_{\sigma(-\Delta)} \left( \int P(\lambda, x, y) f(y) d\mu(y) \right) d\mu(\lambda)
\]
\[-\Delta \int P(\lambda, x, y) f(y) d\mu(y) = \lambda \int P(\lambda, x, y) f(y) d\mu(y)\]
If \( P_\lambda f(x) = \int P(\lambda, x, y) f(y) d\mu(y) \) then \( ||f||_2^2 = \sum_{\lambda \in \sigma(-\Delta)} ||P_\lambda f||_2^2 \), and so we aim at
\[
||f||_2^2 = \int_{\sigma(-\Delta)} ||P_\lambda f||_2^2 d\mu(\lambda).
\]

Our plan:
- find a continuation from graphs to fractafolds.
- find the explicit spectral resolution of the graph Laplacian on \( \Gamma \);
- describe explicitly a Hilbert space of \( \lambda \)-eigenfunctions with norm \( || \cdot ||_\lambda \);

Acknowledgments. We are grateful to Peter Kuchment for very helpful discussions, and to Eugene B. Dynkin for asking questions about the periodic fractal structures.
1. Set-up results for infinite Sierpiński fractafoilds

1.1. Laplacian on the Sierpiński gasket. Let $\Delta_{SG}$ be the Laplacian on $SG$, and $\mu_{SG}$ be the normalized Hausdorff probability measure on $SG$.

![Sierpiński gasket]

**Figure 1.1.** Sierpiński gasket.

Then $\Delta_{SG}$ is self-adjoint on $L^2(SG, \mu_{SG})$ with appropriate boundary conditions and, using Kigami’s resistance (or energy) form,

$$E(f, f) = \lim_{n \to \infty} \left( \frac{5}{3} \right)^n \sum_{x, y \in V_n, x \sim y} (f(x) - f(y))^2 = -\frac{3}{2} \int_{SG} f \Delta_{SG} f \, d\mu_{SG}$$

for functions in the corresponding domain of the Laplacian (Dirichlet or Neumann).
Theorem 1.1. The Laplacian $\Delta$ is self-adjoint and
$$\mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma'_\infty \subset \sigma(-\Delta) \subset \mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma_\infty.$$ Moreover, the spectral decomposition $-\Delta = \int_{\sigma(-\Delta)} \lambda dE(\lambda)$ can be written as
$$-\Delta = \int_{\mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \setminus \Sigma_\infty} \lambda M(\lambda) \Psi^*_\lambda d\left(\mathcal{E}_{\Gamma_0}(\mathcal{R}(\lambda))\right) \Psi_\lambda + \sum_{\lambda \in \Sigma_\infty} \lambda E\{\lambda\}.$$ Here $E\{\lambda\}$ denotes the eigenprojection if $\lambda$ is an eigenvalue. All eigenvalues and eigenfunctions of $\Delta$ can be computed by the spectral decimation method. Furthermore, the Laplacian $\Delta$ on the Sierpiński fractafold $\mathcal{F}$ has the spectral decomposition of the form
$$-\Delta f(x) = \int_{\mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \setminus \Sigma_\infty} \lambda \left(\int_{\mathcal{F}} P(\lambda, x, y) f(y) d\mu(y)\right) dm(\lambda) + \sum_{\lambda \in \Sigma_\infty} \lambda E\{\lambda\} f(x)$$ where $m = m_{\Gamma_0} \circ \mathcal{R}$ and
$$P(\lambda, x, y) = M(\lambda) \sum_{u, v \in V_0} \psi_{v, \lambda}(x) \psi_{u, \lambda}(y) P(\lambda, u, v).$$
1.2. Infinite Sierpiński gaskets.

Figure 1.2. A part of an infinite Sierpiński gasket.
Figure 1.3. An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathcal{R}(\cdot)$, the vertical axis contains the spectrum of $\sigma(-\Delta_{\Gamma_0})$ and the horizontal axis contains the spectrum $\sigma(-\Delta)$.

**Theorem 1.2.** On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a dense set of eigenvalues $\mathcal{R}^{-1}(\Sigma_0)$ of infinite multiplicity and a singularly continuous component of spectral multiplicity one supported on $\mathcal{R}^{-1}(J_R)$. [T98, Quint09]
Problems:
(a) Find an explicit formula for $P_{\lambda}(a, b)$;
(b) Give an explicit description of the projection operator $\tilde{P}_{6}$;
(c) Find an explicit description of $\xi_{\lambda}$ and its inner product, and transfer this to $\tilde{\xi}_{\lambda}$ of $\Gamma_0$.

**Conjecture 1.3.** For $\mu - a.e. \lambda$ there exists a Hilbert space of $\lambda$-eigenfunctions $\xi_{\lambda}$ with inner product $\langle, \rangle_{\lambda}$ such that $P_{\lambda}f \in \xi_{\lambda}$ for $\mu - a.e. \lambda$ for every $f \in \ell^2(\Gamma)$, and

$$\langle P_{\lambda}f, f \rangle = \langle P_{\lambda}f, P_{\lambda}f \rangle_{\lambda}.$$ 

Moreover a similar statement holds for $\langle \tilde{P}_{\lambda}F, F \rangle$. 

2. The Tree Fractafold
We discuss an explicit Plancherel formula on $\Gamma$, given in terms of the modified mean inner product
\[<f, g>_M = \lim_{N \to \infty} \frac{1}{N} \sum_{d(x, x_0) \leq N} f(x)g(x).\]

We deal with eigenspaces for which the limit exists and is independent of the point $x_0$. This is not the usual mean on $\Gamma$, since the cardinality of the ball $\{x : d(x, x_0) \leq N\}$ is $O(2^n)$, but it is tailor made for functions of growth rate $O(2^{-d(x, x_0)/2})$, which is exactly the growth rate of our generalized eigenfunctions.

**Theorem 2.1.** Suppose $f$ has finite support. Then
\[<P_\lambda f, f>_M = 12b(\lambda)^{-1} <P_\lambda f, P_\lambda f>_M\]
and
\[||f||_{\ell^2(\Gamma)}^2 = \int_{\Sigma} <P_\lambda f, P_\lambda f>_M 12b(\lambda)^{-1} d\mu(\lambda).\]

**Theorem 2.2.** Suppose $F$ has finite support on $\Gamma_0$. Then
\[<\tilde{P}_\lambda F, F>_M = 36b(\lambda)^{-1} <\tilde{P}_\lambda F, \tilde{P}_\lambda F>_M\]
and
\[||F||_{\ell^2(\Gamma_0)}^2 = ||\tilde{P}_\theta F||_2^2 + \int_{\Sigma} <\tilde{P}_\lambda F, \tilde{P}_\lambda F>_M 36b(\lambda)^{-1} d\mu(\lambda).\]
Figure 2.1. A part of $\Gamma_1$ with a 5-eigenfunction (values not shown are equal to zero).
Remark 3.1. Note that on a periodic graph, linear combinations of compactly supported eigenfunctions are dense in an eigenspace.

(see [Kuchment05, Theorem 8], [Kuchment93], [KuchmentPost, Lemma 3.5])

The computation of compactly supported 5- and 6-series eigenfunctions is discussed in detail in [St03, T98], and the eigenfunctions with compact support are complete in the corresponding eigenspaces. In particular, [St03, T98] show that any 6-series finitely supported eigenfunction on $\Gamma_{n+1}$ is the continuation of any finitely supported function on $\Gamma_n$, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathcal{F}$ can be computed using the eigenfunction extension map on fractafolds (see Subsection ??). Similarly, any 5-series finitely supported eigenfunction on $\Gamma_{n+1}$ can be described by a cycle of triangles (homology) in $\Gamma_n$, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathcal{F}$ is computed using the eigenfunction extension map on fractafolds.
Example 3.2. *The Ladder Fractafold.*

Figure 3.1. The graphs $\Gamma$ and $\Gamma_0$ for the Ladder Fractafold
Example 3.3. *The Honeycomb Fractafold.*

**Figure 3.2.** A part of the infinite periodic Sierpiński fractafold based on the hexagonal (honeycomb) lattice.
4. Non-fractafold examples

Figure 4.1. A part of the periodic triangular lattice finitely ramified Sierpiński fractal field and the graph $\Gamma_0$. 
Figure 4.2. Computation of the spectrum on the triangular lattice
finitely ramified Sierpiński fractal field.

**Proposition 4.1.** The Laplacian on the periodic triangular lattice finitely ramified
Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum.

The absolutely continuous spectrum is $\mathcal{R}^{-1}[0, \frac{16}{3}]$.

The pure point spectrum consists of two infinite series of eigenvalues of infinite
multiplicity. The series $5\mathcal{R}^{-1}\{3\} \subset \mathcal{R}^{-1}\{6\}$ consists of isolated eigenvalues, and
the series $5\mathcal{R}^{-1}\{5\} = \mathcal{R}^{-1}\{0\}\setminus\{0\}$ is at the gap edges of the a.c. spectrum. The
eigenfunction with compact support are complete in the p.p. spectrum. The spectral
resolution is given in the main theorem.