Limits of self-similar graphs and criticality of the Abelian Sandpile Model

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(G_n, v_n) \xrightarrow{n \to \infty} (G, v) \text{ iff for any } r > 0 \text{ there exists } N \text{ such that for all } n \geq N \text{ the ball } B_{G_n}(v_n, r) \text{ is isomorphic to the ball } B_G(v, r).
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If we are given a sequence \( \{G_n\}_{n \geq 1} \) of graphs (with bounded degrees) without any rooting, we can choose a root in each \( G_n \) uniformly at random. This defines a sequence of probability measures on \( X^* \); its weak limit \( \rho \) is the random weak limit of \( \{G_n\}_{n \geq 1} \) (Benjamini, Schramm).

Thus, \( \rho \) is a probability distribution on the limits of \( \{(G_n, v_n)\}_{n \geq 1} \) in \( X^* \) for all possible choices of roots \( v_n \) in \( G_n \).
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Actions by automorphisms of rooted trees provide many sequences of graphs converging to interesting limits.

Let $\Gamma \leq Aut(T)$ be finitely generated by $S \cup S^{-1}$. For each $n \geq 1$, the Schreier graph $G_n \equiv G(\Gamma, S, \{0, 1\}^n)$ is given by

1. $V(G_n) = \{0, 1\}^n$;
2. $v, w \in V(G_n)$, $v \sim w$ iff $\exists s \in S \cup S^{-1} | s(v) = w$. 

Let $S = \{0, 1\}$ and $S^{-1} = \{\bar{0}, \bar{1}\}$. The graph $G_1$ is the product of the two-element chain with itself, the graph $G_2$ is the product of the two-element chain with the product of the two-element chain with itself, and so on.

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Definition (Grigorchuk, Nekrashevych)

A group $\Gamma$ acting by automorphisms on $T$ is self-similar if, for all $g \in \Gamma$ and $v \in V(T)$, $g|_v \in \Gamma$.
T has a self-similar structure: each $g \in \text{Aut}(T)$ can be written as

$$g = \tau(g|_0, g|_1)$$

where $\tau \in \text{Sym}(2)$ and $g|_0, g|_1 \in \text{Aut}(T)$.

Definition (Grigorchuk, Nekrashevych)

A group $\Gamma$ acting by automorphisms on $T$ is self-similar if, for all $g \in \Gamma$ and $v \in V(T)$, $g|_v \in \Gamma$. 
An interesting example of a self-similar group is the Basilica group $\mathcal{B}$ introduced by Grigorchuk and Żuk ’02. $\mathcal{B}$ is generated by two elements having the following self-similar structure:

$$a = e(b, id) \quad \text{and} \quad b = \tau(a, id).$$
Self-similar graphs
The Abelian Sandpile Model
Criticality of the ASM

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Self-similar graphs
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$G_6$ (as $n \to \infty$).

Figure: The Julia set $J(z^2 - 1)$
If $\Gamma \leq Aut(T)$ acts spherically transitively, then

$$(G_n, \xi_n) \longrightarrow (G_\xi, \xi)$$

for pointed Hausdorff-Gromov convergence, where $G_\xi$ is the infinite orbital Schreier graph of the action of $\Gamma$ on $\partial T$. 
Schreier graphs approximate Julia sets: spectra of Laplacians on Julia sets [Rogers, Teplyaev for Basilica];

Spectra on infinite graphs [Grigorchuk, Šunić];

New examples of random weak limit;

Statistical physics models (Ising model, Potts models, dimer model, Abelian sandpile model,...):

- Usually defined on sequences of finite graphs exhausting an infinite lattice (direct limit).
- What about considering them on covering sequences of Schreier graphs of self-similar groups (projective limit)?
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Depending of $\xi \in \partial T$, $\{(G_n, \xi_n)\}_{n \geq 1}$ converges to various interesting infinite limits.

**Theorem (D’Angeli, Donno, Matter, Nagnibeda, ’10)**

For $i = 1, 2, 4$, let $E_i = \{\xi \in \partial T| (G_{\xi}, \xi) \text{ has } i \text{ ends}\}$. Then,

- $\partial T = E_1 \sqcup E_2 \sqcup E_4$;
- let $\lambda$ be the uniform probability distribution on $\partial T$; then $\lambda(E_1) = 1$;
- there are uncountably many isomorphism classes of one-ended graphs, each of them but one being uncountable;
- there is one isomorphism class of 4-ended graphs (containing one orbit).
The Basilica group
The Abelian Sandpile Model (ASM):

- Self-organized criticality (earthquakes, forest fires) [Bak, Tang, Wiesenfeld, ’88];
- The model is abelian [Dhar, ’90].
A statistical-physics model

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- The model is **abelian** [Dhar, ’90].
Let $G = (V, E)$ be a finite, connected (multi)graph. The model consists in

- configurations $\eta : V \longrightarrow \mathbb{N}$, encoding the repartition of an amount of grains of sand (or chips) on the vertices of $G$;
- local transformation rules between successive configurations: if $\eta(v) \geq \deg(v)$ for some $v \in V$, then $v$ is fired:

$$T_v(\eta)(w) = \begin{cases} 
\eta(w) - \deg(w) & \text{if } v = w, \\
\eta(w) + m(v, w) & \text{if } v \neq w.
\end{cases}$$

where $m(v, w)$ denotes the number of edges between $v$ and $w$.

Introduce one (or more) dissipative vertex $p \in V$; chips reaching $p$ leave the game, ensuring that the firing process eventually stops.

**Abelian Property:** the stable configuration reached through a sequence of firings is independent of the order in which unstable vertices are fired.
Local rules

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Global dynamics

Look at

\{\text{stable configurations}\} \supset R := \{\text{recurrent configurations}\}

Recurrent configurations are characterized by a Markov chain: given a stable configuration \(\eta\),

- drop an extra-chip on a randomly chosen vertex \(v\);
- if the configuration \(\eta + \delta_v\) is unstable, let it stabilize into a new stable configuration \(\eta'\).

The sequence of firings transforming \(\eta + \delta_v\) into \(\eta'\) is an avalanche. By irreducibility, there is a unique stationary distribution \(\mu\) supported by \(R\); \(\mu\) is the uniform distribution.
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Avalanches and criticality

Let $G_1, G_2 \ldots G_n \ldots$ be a sequence of finite graphs converging (in some sense) to an infinite graph $G$.

Consider the random variable $M\text{av}_{G_n} : (\mathcal{R}_n, \mu_n) \to \mathbb{N}$ encoding the mass of the avalanche triggered by adding an extra chip on a fixed vertex $v_n$.

**Definition**

The ASM on $\{G_n\}_{n \geq 1}$ is **critical** if

$$\lim_{n \to \infty} \mathbb{P}_{\mu_n}(M\text{av}_{G_n}(\cdot, v_n) = M) \sim M^{-\delta}$$

for some exponent $\delta > 0$ (called the critical exponent).
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Examples

Many numerical simulations for exhibiting criticality and determining the exponent $\delta$, but few rigorous results.

- **regular infinite tree**: $\delta = 3/2$ [Dhar, Majumdar, ’90];
- $\mathbb{Z}^d$:
  - $d = 1$: not critical;
  - $d = 2$: conjectured that $\delta = 5/4$ [Priezzhev et al., ’96];
  - $d > 4$: conjectured that $\delta = 3/2$ [Priezzhev, ’00];
- Sierpiński gasket: experiments yield $\delta \approx 1.46$ [Kutnjak-Urbanc et al. ’96]
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Given $\xi \in \partial T$, let $\{(G_n, \xi_n)\}_{n \geq 1}$ be the sequence of Schreier graphs of the action of the Basilica group $\mathcal{B}$ on $T$. Then, for almost every $\xi$, the ASM on $\{(G_n, \xi_n)\}_{n \geq 1}$ is critical with critical exponent $\delta = 1$.

Thus, we have exhibited an uncountable family of graphs on which the ASM is critical. Moreover, these graphs are 1-ended, 4-regular and of quadratic growth (properties that they are sharing with $\mathbb{Z}^2$).
ASM on Schreier graphs of the Basilica group

Theorem (Matter, Nagnibeda, ’10)

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Theorem (D’Angeli, Donno, Matter, Nagnibeda, ’10)
There are uncountably many isomorphism classes of two-ended graphs (each of them containing two orbits).

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Suppose that the sequence \( \{(G_n, \xi_n)\}_{n \geq 1} \) of Schreier graphs of the action of \( B \) on \( T \) converges to a two-ended graph. Then, the ASM on \( \{(G_n, \xi_n)\}_{n \geq 1} \) is not critical.

This provides an uncountable family of graphs not quasi-isometric to \( \mathbb{Z} \) on which the ASM is not critical.
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