Non-linear correction to the gyrokinetic polarization density

Alexey Mishchenko, Alain Brizard

Keywords: gyrokinetic theory, Lie transform, quasineutrality equation

Introduction

- **New regimes on the plasma edge: large gradients, strong turbulence, sheared rotation, coherent structures**

- **Gyrokinetic in boundary layers: ITB, pedestal, stellarator root jumping**

- **Multiple-scale physics: system scale, meso-scale (boundary-layer scale), micro-scale (ambient microturbulence)**

- **New features: nonlinear contributions to the polarization density in the full-f approach to the edge plasmas (nonlinearities in distribution function)**

$$n_{pol} = \int \mathrm{d}^6 Z \; \frac{\partial f}{\partial \mu} \; \widetilde{\phi} \sim \phi^2 \, , \quad f = f_0 + \delta f \, , \quad \delta f \sim \phi$$

- **This talk: “meso-scale” nonlinearities in the quasineutrality condition**

Status of art

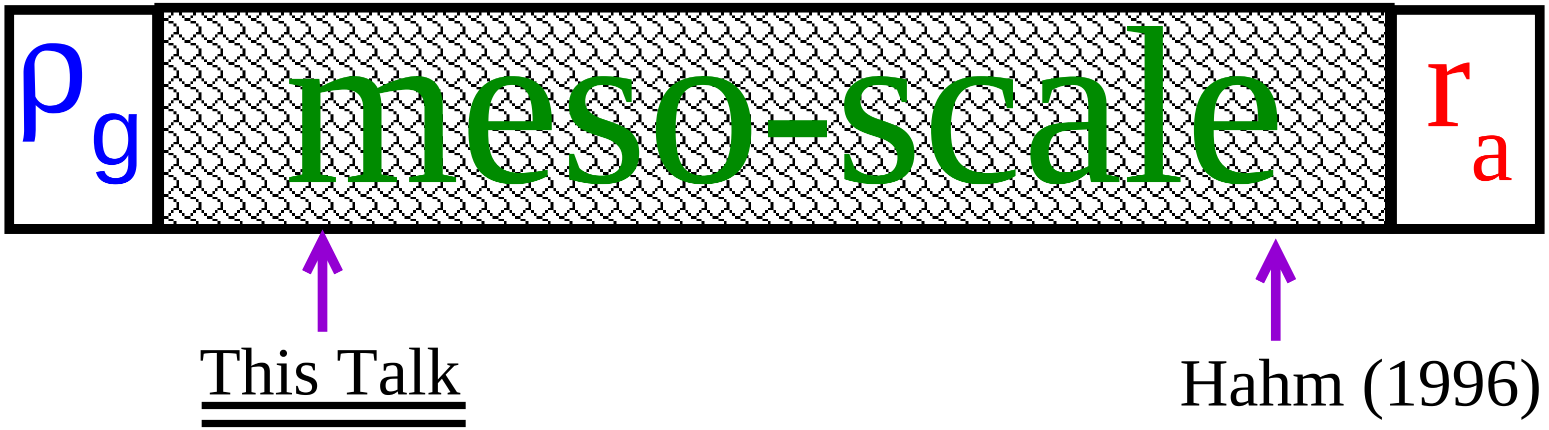
Gyrokinetics with flows (with a background electric field)

$$\Gamma = e\vec{A}^* \cdot \mathrm{d}\vec{X} + \mu \mathrm{d}\theta - H \mathrm{d}t$$
$$H = \frac{m}{2}(v_\parallel \vec{b} + \vec{u}_0)^2 + \mu B + e\Phi + microturbulence$$
$$\vec{A}^* = \vec{A} + \frac{m}{e}(v_\parallel \vec{b} + \vec{u}_0)$$

$\vec{u}_0$: fluid mass velocity (Brizard)
$\vec{u}_0 = \vec{u}_E$: $\vec{E} \times \vec{B}$ velocity (Hahm)
Improvements: time-dependent $\vec{u}_0$ (Miyato), more accurate $\vec{u}_0$ (Kawamura)

Requirement: $\vec{u}_0$ is a slow function of spatial coordinates
Question: what if Φ changes on a scale comparable with gyroradius (not much slower than the gyroradius)?

Characteristic scales



- Mesoscale potential Φ + microturbulence ϕ
- Standard gyrokinetic ordering (microturbulence): $u_E = \nabla\phi/B \sim \epsilon v_{\mathrm{th}}$
- Mesoscale ordering: $U_E = \nabla\Phi/B \sim \sqrt{\epsilon} v_{\mathrm{th}}$ (rotation is strong but subsonic)
- Microturbulence length $l_E \sim \rho_g$
- Mesoscale length L_E can be comparable with ρ_g

“Primitive” Hamiltonian

ExB velocity subsonic $\Rightarrow$ start with $\gamma = e\vec{A}^* \cdot \mathrm{d}\vec{X} + \mu \mathrm{d}\theta - h\mathrm{d}t$ (lab. frame)

$$h = \left(\frac{mv_\parallel^2}{2} + \mu B \right) + \sqrt{\epsilon} e\Phi + \epsilon e\phi$$

Taking $\sqrt{\epsilon}$ as the small parameter for the perturbation theory to be constructed, one can split this Hamiltonian into the parts of different orders:

$$h_0 = \frac{mv_\parallel^2}{2} + \mu B \ , \quad h_1 = e\Phi \ , \quad h_2 = e\phi$$

We expand the particle Hamiltonian $h \equiv h_0 + \sqrt{\epsilon}\, h_1 + \epsilon\, h_2$

The new reduced Hamiltonian is $H \equiv H_0 + \sqrt{\epsilon}\, H_1 + \epsilon\, H_2 + \epsilon^{3/2}\, H_3 + \epsilon^2\, H_4 + \cdots$

Assume $\nabla_\perp \Phi / B \sim \sqrt{\epsilon} v_{\mathrm{th}}$; $\Phi(\vec{R} + \vec{\rho})$ can change on a scale down to ρ_g

Remove gyro-dependencies from the Hamiltonian function (Lie transform)

Hamiltonian Lie transform

$$H = \mathsf{T}_\epsilon^{-1} h \; , \quad h = h_0 + \epsilon h_1 + \epsilon h_2 + \dots$$

$$\mathsf{T}_\epsilon^{-1} \equiv \cdots \; \exp\left(-\,\epsilon^4\,\pounds_4\right)\; \exp\left(-\,\epsilon^3\,\pounds_3\right)\; \exp\left(-\,\epsilon^2\,\pounds_2\right)\; \exp\left(-\,\epsilon\,\pounds_1\right)$$

$$H_0 = h_0, \quad H_1 = h_1 - \pounds_1 h_0,$$
$$H_2 = h_2 - \left(\pounds_2 - \frac{1}{2}\pounds_1^2\right) h_0 - \pounds_1 h_1,$$

<u>NEXT ORDERS ?</u>

Lie transform at higher orders becomes ugly and cumbersome

However: one can use computer algebra systems!

Computer algebra and Lie transform

```
n = 6 ; m = 2$

f(eps) := product(exp(-eps**(n-k)*L[n-k]), k,0,n-1) * sum(h[i]*eps**i, i,0,m)$

H[i] := coeff(taylor(f(eps),eps,0,n),eps,i)$

ratvars(h[0],h[1],h[2])$

rat(H[6]);

kill(all)$
```

Computer algebra in action. Example.

ami@dhcp-host6: $ maxima -b LieTransform.mxm

$$H_6 = -\frac{\left(720\,L_4 - 720\,L_1\,L_3 - 360\,L_2^2 + 360\,L_1^2\,L_2 - 30\,L_1^4\right)}{720}\,h_2 +$$
$$\frac{\left(720\,L_5 - 720\,L_1\,L_4 + \left(-720\,L_2 + 360\,L_1^2\right)\,L_3 + 360\,L_1\,L_2^2 - 120\,L_1^3\,L_2 + 6\,L_1^5\right)}{720}\,h_1$$
$$+\frac{720\,L_6 - 720\,L_1\,L_5 + \left(-720\,L_2 + 360\,L_1^2\right)\,L_4 - 360\,L_3^2}{720}\,h_0 +$$
$$\frac{\left(720\,L_1\,L_2 - 120\,L_1^3\right)\,L_3 + 120\,L_2^3 - 180\,L_1^2\,L_2^2 + 30\,L_1^4\,L_2 - L_1^6}{720}\,h_0$$

Hamiltonian Lie transform

- **Lie Transform:** $H = T_\epsilon^{-1} h, \quad T_\epsilon^{-1} = \dots T_3^{-1} T_2^{-1} T_1^{-1} = \dots e^{-\epsilon^3 L_3} e^{-\epsilon^2 L_2} e^{-\epsilon L_1}$

- **Equations for the generating functions** (use computer algebra!)

$$\hat{D}_0 S_1 = H_1 - h_1 \ , \quad L_1 = \{S_1, \cdot\}$$
$$\hat{D}_0 S_2 = 2\,(H_2 - h_2) - \left(L_1 H_1 + T_1^{-1} h_1\right) \ , \quad L_2 = \{S_2, \cdot\}$$
$$\hat{D}_0 S_3 = 3\,(H_3 - h_3) - \left(L_1 H_2 + L_2 H_1 + T_2^{-1} h_1 + 2 T_1^{-1} h_2\right)$$
$$\hat{D}_0 S_4 = 4\,(H_4 - h_4) - \left(L_1 H_3 + L_2 H_2 + L_3 H_1 + T_3^{-1} h_1 + 2 T_2^{-1} h_2 + 3 T_1^{-1} h_3\right)$$

- **Lie transform at higher orders** (computer algebra helps here!)

$$T_0 = T_0^{-1} = 1 \ , \quad T_1^{-1} = L_1 T_0^{-1} \ , \quad T_2^{-1} = \frac{1}{2}\left(L_2 T_0^{-1} + L_1 T_1^{-1}\right)$$
$$T_3^{-1} = \frac{1}{3}\left(L_3 T_0^{-1} + L_2 T_1^{-1} + L_1 T_2^{-1}\right)$$
$$T_4^{-1} = \frac{1}{4}\left(L_4 T_0^{-1} + L_3 T_1^{-1} + L_2 T_2^{-1} + L_1 T_3^{-1}\right)$$

Transformed Hamiltonian

$$H_0 = h_0$$

$$H_1 = e\langle\Phi\rangle$$

$$H_2 = e\left[\langle\phi\rangle + \underbrace{\frac{1}{2}\left\langle\{S_1, \widetilde{\Phi}\}\right\rangle}_{polarization\ \ density}\right]$$

$$H_3 = \frac{e}{3!}\left\langle\{S_1,\{S_1,\Phi\}\} + \{S_2,\widetilde{\Phi}\} + 4\{S_1,\widetilde{\phi}\}\right\rangle$$

$$H_4 = \frac{e}{4!}\Big\langle\{S_1,\{S_2,\Phi\}\} + 2\{S_2,\{S_1,\Phi\}\} + 2\{S_3,\widetilde{\Phi}\} + \\ + \underbrace{6\{S_2,\widetilde{\phi}\}}_{polarization\ \ density} + 6\{S_1,\{S_1,\phi\}\} + \{S_1,\{S_1,\{S_1,\Phi\}\}\}\Big\rangle$$

Solve equations for the generating functions and find the Hamiltonians

Second-order gyrokinetic Hamiltonian

Use multiple-scale approach to solve the equations for generating functions

$$\hat{D}_0 S_1 = \frac{\partial S_1}{\partial t} + \{S_1, h_0\} = \underbrace{\frac{\partial S_1}{\partial t}}_{\sim \Omega} + \underbrace{v_\parallel \nabla_\parallel S_1}_{\sim K_\parallel v_\parallel} + \underbrace{\vec{v}_d \cdot \nabla S_1}_{\sim K_\perp v_\mathrm{d}} + \underbrace{\omega_\mathrm{c} \frac{\partial S_1}{\partial \theta}}_{\sim \omega_\mathrm{c}} = -e\widetilde{\Phi}$$

$$S_1 = -\sqrt{\epsilon}\,\frac{e\widetilde{\Phi}^{(1)}}{\omega_c} + \epsilon^{3/2}\,\frac{v_\parallel \nabla_\parallel + \vec{v}_d \cdot \nabla}{\omega_c}\left(\frac{e\widetilde{\Phi}^{(2)}}{\omega_c}\right)$$

Substituting this solution, one can write with accuracy up to ϵ^2:

$$H_2 = \epsilon\left[e\langle\phi\rangle - \frac{e^2}{2B}\left(\underbrace{\frac{\partial\langle\widetilde{\Phi}^2\rangle}{\partial\mu}}_{pol.\ density} + \frac{1}{e\omega_\mathrm{c}^*}\left\langle \nabla\widetilde{\Phi}^{(1)} \cdot (\vec{b} \times \nabla\widetilde{\Phi})\right\rangle\right)\right]$$

$$+\,\epsilon^2\,\frac{e^2}{2B}\,\frac{\partial}{\partial\mu}\left\langle\widetilde{\Phi}\ \ (v_\parallel \nabla_\parallel + \vec{v}_d \cdot \nabla)\left(\frac{\widetilde{\Phi}^{(1)}}{\omega_\mathrm{c}}\right)\right\rangle$$

Third-order gyrokinetic Hamiltonian

$$\epsilon\left(\frac{\partial S_2}{\partial t}+v_\parallel\nabla_\parallel S_2+\vec{v}_d\cdot\nabla S_2\right)+\omega_c\frac{\partial S_2}{\partial\theta}=$$
$$-\,\epsilon\left(2e\widetilde{\phi}+e\left[2\{S_1^{(0)},\langle\Phi\rangle\}+\{S_1^{(0)},\widetilde{\Phi}\}-\left\langle\{S_1^{(0)},\widetilde{\Phi}\}\right\rangle\right]\right)-$$
$$-\,\epsilon^2\,e\left[2\{S_1^{(2)},\langle\Phi\rangle\}+\{S_1^{(2)},\widetilde{\Phi}\}-\left\langle\{S_1^{(2)},\widetilde{\Phi}\}\right\rangle\right]$$

Here, we have already anticipated that S_2 has the perpendicular spatial scales down to ρ_i and the characteristic frequency $\omega\sim\epsilon\,\omega_{ci}$. Employing a subsidiary multiple-scale expansion:

$$S_2=\epsilon\left(S_2^{(0)}+\sqrt{\epsilon}\,S_2^{(1)}+\epsilon\,S_2^{(2)}+\ldots\right)\qquad(1)$$

we can solve the equation for S_2.

Third-order gyrokinetic Hamiltonian

$$S_2^{(0)} = -\frac{2\,e\,\widetilde{\phi}^{(1)}}{\omega_c} - \frac{e}{\omega_c}\int\limits_0^\theta \mathrm{d}\theta' \left[2\{S_1^{(0)}, \langle\Phi\rangle\} + \left(\{S_1^{(0)}, \widetilde{\Phi}\} - \left\langle\{S_1^{(0)}, \widetilde{\Phi}\}\right\rangle\right)\right] \quad (2)$$

The following notations have been introduced:

$$\widetilde{\phi}^{(0)} = \widetilde{\phi}\ , \quad \widetilde{\phi}^{(n)} = \int\limits_0^\theta \mathrm{d}\theta'\ \widetilde{\phi}^{(n-1)}(\theta')\ , \quad n = 1, 2, \ldots \quad (3)$$

Using the relation $\widetilde{\Phi}\{\widetilde{\Phi}^{(1)}, \widetilde{\Phi}\} = 1/2\,\{\widetilde{\Phi}^{(1)}, \widetilde{\Phi}^2\}$, one obtains the third-order gyrokinetic Hamiltonian:

$$H_3 = \epsilon^{3/2}\frac{e^2}{B}\frac{\partial}{\partial\mu}\left[-\left\langle\widetilde{\Phi}\,\widetilde{\phi}\right\rangle + \frac{e}{2B}\left\langle\widetilde{\Phi}^2\right\rangle\frac{\partial\langle\Phi\rangle}{\partial\mu} + \frac{e}{6B}\frac{\partial}{\partial\mu}\left\langle\widetilde{\Phi}^3\right\rangle\right] \quad (4)$$

Variational principle (Brizard)

The gyrokinetic electrostatic action functional in the extended phase space

$$\mathcal{A}_{\rm gy} = -\sum_{s=i,e,\dots}\int (H[Z,\Phi,\phi]-w)\,\mathcal{F}(Z)\,\mathrm{d}^8 Z$$

Here, $\mathcal{F}(Z) = f(Z)\delta(H-w)$ is the distribution function in the extended phase space, $f(Z)$ is the usual gyrokinetic distribution function and w is the phase-space coordinate conjugate to time. The gyrokinetic field equation is then given by the variational derivative:

$$\delta\mathcal{A}_{\rm gy}/\delta\Phi(\vec{x}) = 0$$

Quasineutrality equation

Neglecting, as usual, the Debye screening, we obtain the gyrokinetic quasineutrality equation for the microturbulence potential ϕ as follows:

$$\sum_{s=i,e,\dots} e \int \Big\{ f + \frac{e}{B} (\widetilde{\Phi} + \widetilde{\phi}) \frac{\partial f}{\partial \mu} +$$

$$+ \frac{e^2}{2B^2} \left[\widetilde{\Phi}^2 \frac{\partial^2 f}{\partial \mu^2} + \left(\frac{\partial \langle \Phi^2 \rangle}{\partial \mu} - 2\,\Phi\,\frac{\partial \langle \Phi \rangle}{\partial \mu} \right) \frac{\partial f}{\partial \mu} \right] \Big\} \, \delta(\vec{R} + \rho - \vec{x}) \, \mathrm{d}^6 Z = 0$$

In the long-wavelength approximation, one can write the quasineutrality equation in the simple form [$\bar{n}_s$ is the usual gyrokinetic density and $\Phi = \Phi(\vec{x})$]:

$$- \nabla_\perp \left[\frac{n_i}{B\omega_c} \nabla_\perp (\phi + \Phi) \right] - \frac{1}{2} \nabla_\perp \left[\nabla_\perp \left(\frac{n_i}{B^2 \omega_c^2} \right) (\nabla_\perp \Phi)^2 \right] = n_i - n_e$$

One sees that the resulting equation contains the polarization density due to the background electrostatic potential, the polarization density caused by the microturbulence and an additional term which is quadratic in Φ. This term is of the same order as the microturbulence polarization density ($\sim \nabla_\perp^2 \phi$).

Quasineutrality equation

Introduce the notation: $\vec{\kappa}_n = \nabla_\perp n_i / n_i$ **(density gradient)**

$$\nabla_\perp \cdot \underbrace{\left(\frac{n_i \vec{u}_E}{\omega_c} \right)}_{\text{"}momentum\text{"}} - \nabla_\perp \cdot \underbrace{\left(\frac{n_i u_E^2}{2\,\omega_c^2} \; \vec{\kappa}_n \right)}_{\text{"}energy\ \ flux\text{"}} = n_i - n_e$$

$$\vec{u}_E = - \frac{\nabla_\perp \Phi}{B}$$

define the polarization vector $\vec{P}_E = -\nabla_\perp \Phi / (B \omega_c)$

normalize the density to the total ion density n_i

$$\nabla_\perp \cdot \left(\vec{P}_E - \frac{1}{2} P_E^2 \; \vec{\kappa}_n \right) = \delta n$$

Gyrokinetic Vlasov equation and particle energy

Constrained eulerian variation of the distribution function leads to the gyrokinetic Vlasov equation:

$$\frac{\partial f}{\partial t} + \{f, H\} = 0 \tag{5}$$

In practice, the gyrokinetic Hamiltonian function can be taken in a simplified from:

$$H = \frac{mv_\parallel^2}{2} + \mu B + e\langle \Phi + \phi \rangle - \frac{e^2}{2B}\frac{\partial \langle \widetilde{\Phi}^2 \rangle}{\partial \mu} + H_3$$

Here, the ponderomotive energy associated with the background electric field is of the same order as the energy of the microturbulence $e\langle\phi\rangle$.

$$H = \frac{mv_\parallel^2}{2} + \mu B + e(\Phi + \phi) - \frac{mu_E^2}{2} + \ldots$$

$$\vec{u}_E = -\frac{\nabla_\perp \Phi}{B}, \quad \rho_{\rm th} = v_{\rm th}/\omega_c$$

Equations of the gyrocenter motion

- **Gyrocenter motion (parallel streaming + drifts)**

$$\dot{\vec{R}} = v_{\|}\vec{b}^* + \underbrace{\frac{\vec{b}\times\mu\nabla B}{eB_{\|}^*}}_{charge\ \ separation} + \frac{\vec{E}\times\vec{b}}{B_{\|}^*} + \underbrace{\frac{e}{2m\omega_c^2}\frac{\nabla\vec{E}^2\times\vec{b}}{B_{\|}^*}}_{charge\ \ separation\ \ (ponderomotive\ \ drift)}$$

- **Parallel force (including ponderomotive parallel acceleration):**

$$m\dot{v}_{\|} = \underbrace{-\mu\vec{b}^*\cdot\nabla B}_{mirror\ \ force} + \underbrace{e\vec{b}^*\cdot\vec{E}}_{parallel\ \ electric\ \ force} + \underbrace{\frac{e^2}{2m\omega_c^2}\vec{b}^*\cdot\nabla\vec{E}^2}_{ponderomotive\ \ parallel\ \ force}$$

- **Ponderomotive force leads to an additional drift and parallel acceleration**

Summary

- New regimes on the plasma edge (coherent structures, interfaces)
- Meso-scale physics needs "meso-scale gyrokinetics"
- Nonlinear "meso-scale" contributions to quasineutrality condition
- Ponderomotive contribution to the gyrokinetic particle energy and motion
- Higher-order terms for transport of momentum in tokamaks? (Felix)

THANK YOU!