Stochastic methods in palaeoclimates

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Workshop on ‘Stochastic Methods in Climate Modelling, Isaac Newton Institute, Cambridge, 24th August 2010
This work is part of a project funded by the European Research Council:

**Integrating Theory and Observations of the Pleistocene.**

This project, along other Belgian grants, supports a team of five scientists:

**Michel Crucifix** (manager)

**Bernard De Saedeleer** (dynamical systems, stability analysis, synchronisation)

**David Garcia-Alvarez** (stochastic dynamical systems, calibration on summary statistics)

**Guillaume Lenoir** (wavelets, bifurcation analysis)

**Nabila Bounceur** (experiment plans with numerical simulators, emulators)

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Introduction:

- Why a mathematical theory of palaeoclimates
- Stochastic differential equations
  - deterministic paradigms
  - the stochastic resonance
  - relaxation oscillators
- Stochastic methods for model selection and calibration
Greenland $\delta^{18}O$

$\delta^{18}$-O Benthic Stack (roughly ice volume)

Antarctic Dome C $\delta D$

Time (thousand years)

Sources: NGRIP, 2004; Lisiecki and Raymo, 2005, EPICA, 2006. All available on NOAA database.
“Understanding palaeoclimates”

- **Phenomenological**
  - connexion between different components of climate system
  - ‘leads and lags’
  - confrontation of climate records

- **Connexion with the laws of physics (e.g.: fluid dynamics)**
  - use of simulators

- **Dynamical understanding**
  - ‘climate attractors’
  - bifurcation analysis
Astronomical forcing

- obliquity, precession, eccentricity
- changes in distribution of incoming solar radiation
- definite drive on ice mass balance, possibly (many) other means of actions (climate astronomical sensors)
the myth of the ‘sharp spectral peaks’
Insolation contains a rich spectrum

Spectrum Incoming Solar Radiation at Summer Solstice at 65° N

- **Obliquity terms**
- **Precession terms**

“20-ka”

“40-ka”

Amplitude (W/m²) vs. Angular velocity (degree per thousand years)

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**Source:** A. L. Berger, Long-term variations of daily insolation and Quaternary climatic changes, J. Atmos. Sci., 35, 2362–2367 1978

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Benthic record ODP 1143 and its continuous wavelet transform

delta 18-O

Depth (m)

Period

50
100
150

10

2.0
2.5
3.0
3.5
4.0
4.5

0.1
0.2
0.3
0.4

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Which stochastic methods?

- Parameterisation of uncertainties
  - (a parameter may be constant but stochastic)
- Parameterisation of unresolved processes
  - e.g. : Gaussian noise
- Generation of chronologies

Where is theory needed?

- Dynamical understanding of Stochastic Climate models
- Calibration of uncertain parameters and model selection
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Stochastic methods for model selection and calibration
Deterministic paradigm: climate driven by an effective potential function

- **1-d model**

\[ dX = - \frac{d\phi(X, F')}{dX} dt \]

trivial example: \[ \phi(x, t) = \alpha X^2 + F(t)X \]

\[ dX = -(2\alpha X + F')dt \]

linear response model
Feedbacks $\rightarrow$ 2-state models


$$\phi = -\frac{X^4}{4} + X^2 - FX$$

see : recently, P. Ditlevsen, Paleoceanography (2009)
Justification of the 2-state paradigm

- At least one ice-sheet / climate simulator was found to roughly obey this simple equation (LLN - 2 D model)
- Many components of the Earth system may be driven by an equation of this kind
  - Atlantic ocean deep circulation (Stommel [1960])
  - Vegetation models (e.g.: Brovkin et al. [2003])
reproduce the LLN–2D simulator with a 2–state model


fit with the 1-D model: M. Crucifix, manuscript in revision for ‘The Holocene’


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Deterministic paradigm #2: interaction between component with different time-scales

\[
\frac{dX}{dt} = \frac{1}{\tau_x} f(X, Y, F(t)) \\
\frac{dY}{dt} = \frac{1}{\tau_y} g(X, Y, F(t)), \text{ with } \tau_y \ll \tau_x
\]
A much-studied mathematical object

Liénard form:

\[ f = -Y \]
\[ g = -\left(\phi'(Y) - X\right) \]

Van–der–Pol Bonhoeffer form:

\[ f = -(Y + \beta \ (F(t))) \]
\[ g = -\left(\phi'(Y) - X\right), \quad \phi'(Y) = \frac{Y^3}{3} - Y \]

‘Self-sustained’ oscillations

Excitable system for ‘beta’ at the fringe of the Hopf bifurcation

Synchronisation (phase-locking) on F
Fitting an astronomically driven vdp-oscillator to palaeoclimate data is easy.
The hidden state model as a surrogate for the slow–fast system

\[
\frac{1}{\tau_x} \frac{dX}{dt} = f^g(X, Y, F(t))
\]

\[g = \text{a discrete process}\]

\[g = 0 \quad \text{during the ice build up process}\]

\[g = 1 \quad \text{during deglaciation}\]

see Guckenheimer et al. (2003), SIAM, 2, 1–35 (Jump oscillator)
see recent works by Peavoy and Franzke in C. Past. Discussion for hidden-state model in the context of a palaeoclimate model-selection problem

Possibility of more time scales interacting (e.g.: turbulence involves infinite time scales)

\[
\frac{1}{\tau_x} \frac{dX}{dt} = f(X, Y, Z, \ldots, F(t)) \\
\frac{1}{\tau_y} \frac{dY}{dt} = g(X, Y, Z, \ldots, F(t)) \\
\frac{1}{\tau_z} \frac{dZ}{dt} = h(X, Y, Z, \ldots, F(t)) \\
\ldots
\]

- separation of time scales = average out z
- stochastic parameterisation = represent z as a stochastic process
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Stochastic methods for model selection and calibration
Langevin equation with 2–well potential

\[ dX = -\frac{d\phi(X, F)}{dX} dt + \sigma^{1/2} d\omega \]

\[ \phi = -\frac{X^4}{4} + X^2 - FX \]

- Suppose $F$ periodic, but with weak amplitude
- Carefully chosen noise amplitude will with catalyse jumps between the two potential wells
- $\rightarrow$ strong periodic component in the system response
Fig. 7. Peaks of power spectra of numerical solutions versus $\epsilon$. 
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Stochastic methods for model selection and calibration
Effect of stochastic perturbation on astronomically-driven vdp: two sample paths
so what happened?
Let’s go back to the deterministic system

work to be submitted:
B. De Saedeleer, M. Crucifix and S. Wieczorek
Autonomous oscillator: neutral with respect to the phase ($\lambda \text{ max} = 0$)

Different initial conditions
Driven oscillator: converge to ‘attracting phases’ (λ max < 0)
with astronomical forcing: three ‘clusters’ (depends on parameters)
One may define the ‘pullback attractor’ and the ‘basin of attraction’
Periodic forcing (here: 2:1 locking) pullback attractor

Astronomical forcing

Effect of stochastic perturbation on astronomically-driven vdp: two sample paths

transitions between two pullback attractors
Partial summary

- **1-D systems, with 2-well potentials:**
  - noise may induce stochastic resonance: amplifies an external forcing (but amplitude needs to be right)
  - Well known and documented result

- **2-D Oscillators**
  - noise creates a probability of transition between different ‘pullback’ attractors
  - probability depends on noise amplitude, forcing and dynamics
  - ... problem to be further explored.
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Stochastic methods for model selection and calibration
Reducing uncertainties and select models

OUR PROBLEMS

- Explore climate’s structural behaviour
  - shape of potential functions
  - existence of links between different components
- Estimate past climate state
- Estimate age of palaeoclimate records
motivating example: does climatic precession drive CO2?

Ruddiman (2003)

Tilt

Precession

Forcing

Feedback

\[ \text{ICE} \]

\[ \text{CO}_2 \]

\[ \text{CH}_4 \]

\(~ 6500 \text{ yrs}~\)

\(~ 4500 \text{ yrs}~\)

\[ \text{CH}_4 \] \[ \text{CO}_2 \]


Model seen as a ‘data generating process’

Simulators (e.g. : GCM) → “Model” → Observations

“Model” → Physical arguments

“Model” → Phenomenological arguments
In this approach the model is not immediately inferred from observations (this is a myth: there are always underlying hypotheses), but it may be refined by a process of model selection.
As much as possible: find an embedded model framework

\[ \tau_x dX = - (cY + \beta + \alpha F(t)) dt + \sigma d\omega \]
\[ \tau_x dY = - (\phi(Y; \theta) - X) dt \]

e.g.:

if \( c = 0 \): linear Langevin model
if \( \alpha = 0 \): no forcing
etc.
... three years of ruminations

- **Metropolis - Hastings methods**
  - application to palaeoclimate by Hargreaves and Annan [2002] for deterministic case
  - remember: stochastic cause probabilistic phase-slips
  - need refinements. Proposal by Andrieux et al. [2010].

- **Particle filter for parameter and state estimation (Liu and West, 2001)**
  - (implementation on an R package that we called APF1step)
  - Bayesian framework
One application (red = stochastic)

\[ \tau_x dX = (-Y + \beta + \alpha F(t))dt + \sigma d\omega \]
\[ \tau_y dY = -(\phi(Y) - X)dt \]

\[ \tau_x \sim 10 \text{kyr} \cdot \ln N(0, 1) \]
\[ \tau_y \sim 10 \text{kyr} \cdot \ln N(0, 1) \]
\[ c, \beta \sim \mathcal{N}(0, 1) \]
Filter: castastrophe at ‘terminations’

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weather model

d18O LR04

beta

LnCt

time (kyr)


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Filtered parameters no longer give rise to oscillating behaviour
... three years of ruminations

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- **Particle filter for parameter and state estimation**
  - (implementation on an R package that we called APF1step)
  - Bayesian framework
  - proved inefficient for slow-fast systems

- **Consider summary statistics**
  - implementation: Approximate Bayesian Computation
Inferences about a ‘data generating process’

- Observations
- Simulators (e.g.: GCM)
- “Model”
- Physical arguments
- Phenomenological arguments
- Summary statistic

The ideal summary statistic:

- concentrates on what you want to learn
- insensitive or little sensitive to nuisance parameters

If you want to learn about the dynamics

- try to build summary statistic not too sensitive to time-scale errors

... and vice versa: if you want to learn about the time-scale ...

- running project: wavelet-based summary statistic (PhD student: Guillaume Lenoir)
Summary

- **Why a mathematical theory today?**
  - still lack of dynamical understanding
  - connexion with statistics to rationalise model selection

- **Stochastic parameterisations**
  - represent ‘fast modes’
  - in oscillators: catalyse phase slips: unpredictability
  - great scope for sophistication: e.g.: multiplicative noise

- **Methods for model selection and calibration**
  - Filtering approaches as they are unsuitable for slow–fast systems
  - Importance of finding appropriate summary statistics