Rate-dependant Tipping Points:
The Example of the Compost-Bomb Instability

Sebastian Wieczorek

Mathematics Research Institute
University of Exeter, UK

Prof. Peter Ashwin
Mrs. Catherine Luke
Prof. Peter Cox

Stochastic Methods in Climate Modelling, Cambridge, August 25, 2010
Outline:

1. The Compost-Bomb Instability: Motivation

2. Excitability
   - State Jumps
   - Ramped Parameter

3. The Compost-Bomb Instability: Explanation

4. Conclusions
United Nations Framework Convention on Climate Change (UNFCCC)

“The ultimate objective is... stabilisation of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system...

... such a level should be achieved within a time frame sufficient to allow ecosystems to adapt naturally to climate change, to ensure food production is not threatened and to enable economic development to proceed in a sustainable manner”

dangerous levels and dangerous rates
**Facts:**
Peatland soils contain 400–1000 billion tones of carbon. Spontaneous combustion of compost heaps is well known!

**Question:**
How will peatlands respond to global warming?

**Climate model (coupled PDEs)**

**Full–scale Climate model (coupled PDEs)**

**Low–dimensional simplified model (coupled ODEs)**
Simple Soil-Carbon-Temperature Model

\[
\frac{d}{dt} C = \Pi - C \ r(T), \\
\frac{d}{dt} T = -\frac{\lambda}{\mu} (T - T_a) + \frac{A}{\mu} C \ r(T), \\
\frac{d}{dt} T_a = v.
\]

\(C\)—soil carbon content, \(T\)—soil temperature, \(T_a\)—atmospheric temperature

\(\Pi\)—litter fall from plants

\(r(T) = r_0e^{\alpha T}\) — microbial decomposition

\(\lambda\)—soil-to-atmosphere heat transfer

\(\mu\)—thermal inertia

\(A\)—constant of proportionality

\(v\)—rate of global warming

Stability For Fixed $T_a$ ($v = 0$)

Globally stable equilibrium

$$a = (C^{eq}, T^{eq}) = (\frac{\Pi}{r(T^{eq})}, \quad T_a + A\Pi/\lambda)$$

No dangerous level!
Effects of Global Warming
Compost-Bomb Instability

\[ \nu = 7.5 \, ^\circ C/100 \, \text{years} \]

[\nu = 9 \, ^\circ C/100 \, \text{years}]

Soil temperature \( T (\, ^\circ C) \)

Time (years)

[\text{globally stable static equilib.}]

Outline:

1. The Compost-Bomb Instability: Motivation

2. Excitability
   - State Jumps
   - Ramped Parameter

3. The Compost-Bomb Instability: Explanation

4. Conclusions
Excitability Properties

(P1) A (stable) quiescent state

(P2) A threshold for excitable response

(P3) A return mechanism that:

• specifies the “shape” of excitable response and
• brings the system back to the quiescent state
Excitability Types

\[ \dot{x} = f(x, p), \quad x \in \mathbb{R}^n, \quad n \geq 2 \]

\[ \dot{z} = g(x, z, p, \epsilon), \quad 0 < \epsilon \ll 1 \]

*a saddle equilibrium*

folded slow (critical) manifold

green: slow manifold
Outline:

1. The Compost-Bomb Instability: Motivation

2. Excitability
   • State Jumps
   • Ramped Parameter

3. The Compost-Bomb Instability: Explanation

4. Conclusions
The General Problem: A Ramped Excitable System

**Ramping:** a slow and monotonic change in one of the system parameters $pr$

\[
\frac{d}{dt} x = f(x, z, pr, p, \epsilon)
\]

\[
\epsilon \frac{d}{dt} z = g(x, z, pr, p, \epsilon)
\]

\[
\frac{d}{dt} pr = v
\]

**Assumption:** The unramped ($v = 0$) system has a globally stable equilibrium $a$ and a fold $L$ for all static settings of $pr$.

**Questions:**

What is the excitability threshold for $v > 0$?

What is the critical ramping rate $v_c$?
The Phase Space of the General Problem

Slow (critical) manifold $S = S_a \cup L \cup S_r$ is a graph over $p_r$ and $z$: $x = h(z, p_r, p)$

Fold $L$ is given by $\frac{\partial g}{\partial z}|_S = 0$

No equilibrium points for $v > 0$
Evolution of the Fast Variable on the Slow Manifold \( S \)

\[ \epsilon \to 0 \]

Start with the general problem:

\[
\frac{d}{dt} x = f(x, z, pr, p, \epsilon), \quad \epsilon \frac{d}{dt} z = g(x, z, pr, p, \epsilon), \quad \frac{d}{dt} pr = v
\]

Set \( \epsilon = 0 \), differentiate \( dg/dt \), project onto \( S \) using \( x = h(z, pr, p) \)

to get the reduced system:

\[
\frac{d}{dt} z = -\frac{\partial g/\partial x|_S f|_S + \partial g/\partial pr|_S v}{\partial g/\partial z|_S}
\]

\[
\frac{d}{dt} pr = v
\]

that is \textbf{singular} at the fold \( L \), where \( \partial g/\partial z|_S = 0 \), except for special points called \textbf{folded singularities} \( F \), where \( \partial g/\partial x|_S f|_S + \partial g/\partial pr|_S v = 0 \).
Desingularisation

The analysis is greatly facilitated using a clever scaling:

\[ t = -\hat{t} \frac{\partial g}{\partial x}|_S \]

which preserves the direction of time on \( S_a \) but reverses it on \( S_r \). This gives the desingularised system:

\[
\begin{align*}
\frac{d}{d\hat{t}} z &= \frac{\partial g}{\partial x}|_S f|_S + \frac{\partial g}{\partial p_r}|_S v \\
\frac{d}{d\hat{t}} p_r &= -v \frac{\partial g}{\partial z}|_S 
\end{align*}
\]

whose regular equilibrium is a folded singularity \( F \) of the reduced system.
Possible Phase Portraits of the Reduced System (without folded singularities)
Possible Phase Portraits of the Reduced System
(with folded singularities $F$)

- Folded saddle
- Folded saddle-node
- Folded saddle-node
- Folded (stable) focus
- Folded centre
- Folded (unstable) focus
- Folded (stable) node
- Folded (stable) deg. node
The critical rate of ramping $v_c$

Initial condition: $(z^0, p^0_r)$

Folded saddle: $F(v) = (z^F(v), p^F_r(v))$

Stable eigenvector: $[w_1(v), w_2(v)]^T$

The critical rate $v_c$ is the value of $v$ where the singular canard crosses $(z^0, p^0_r)$

$$p^0_r - p^F_r(v_c) = \frac{w_1(v_c)}{w_2(v_c)} \left[ z^0 - z^F(v_c) \right]$$
Solution to the General Problem

A ramped system with folded slow (critical) manifold:

\[
\frac{d}{dt} x = f(x, z, p_r, p, \epsilon), \quad \epsilon \frac{d}{dt} z = g(x, z, p_r, p, \epsilon), \quad \frac{d}{dt} p_r = v,
\]

that preserves a stable equilibrium is excitable if the reduced system:

\[
\frac{d}{dt} z = -\frac{\partial g/\partial x}{\partial g/\partial z} f + \frac{\partial g/\partial p_r}{\partial g/\partial z} v, \quad \frac{d}{dt} p_s = v,
\]

has a folded saddle singularity.

The excitability threshold is related to the singular canard via folded saddle. The critical rate \(v_c\) can be calculated from:

\[
p_r^0 - p_r^F(v_c) = \frac{w_1(v_c)}{w_2(v_c)} \left[ z^0 - z^F(v_c) \right].
\]
Outline:

1. The Compost-Bomb Instability: Motivation

2. Excitability
   - State Jumps
   - Ramped Parameter

3. The Compost-Bomb Instability: Explanation

4. Conclusions
Back to the Soil-Carbon-Temperature Model

\[
\frac{d}{dt} C = \Pi - C \, r_0 \, e^{\alpha T},
\]

\[
\epsilon \frac{d}{dt} T = -\frac{\lambda}{A} (T - T_a) + C \, r_0 \, e^{\alpha T}, \quad \text{where} \quad \epsilon = \frac{\mu}{A} = 0.064
\]

\[
\frac{d}{dt} T_a = v.
\]

Using the general condition get the critical rate of global warming

\[
v_c = \frac{r_0 (1 - \alpha A \Pi / \lambda)}{\alpha} \exp \left( \alpha T_a^0 + 1 \right) \approx 8 \, ^\circ \text{C}/100 \text{ years}
\]
Understanding the Compost-Bomb Instability

\[ v = 7.5 \, ^\circ \text{C}/100 \text{ years} < v_c \]

\[ v = 9.0 \, ^\circ \text{C}/100 \text{ years} > v_c \]
Understanding the Compost-Bomb Instability

\[ v = 7.5 \, ^\circ\text{C}/100\, \text{years} < v_c \]

\[ v = 9.0 \, ^\circ\text{C}/100\, \text{years} > v_c \]
Multiple-Spike Excitable response

\[ v = 30 \, ^\circ \text{C}/100 \, \text{years} > v_c \]
Conclusions:

1. Tipping points—not just bifurcations

2. Rate-dependant tipping points
   as a novel excitability type:
   - Near a saddle
   - Near a folded slow manifold
   - There are more possibilities (multistable systems)
   - Prediction/warning: calculate the critical rate

3. Explained the “compost-bomb instability”:
   - A potential climate tipping point
     that does not involve any bifurcations

Excitability Threshold

\[ \epsilon = 0 \]

\[ 0 < \epsilon \ll 1 \]
A Special Return Mechanism