Using simple models and rigorous mathematics to improve operational atmosphere and ocean modelling

Mike Cullen, Newton Institute programme on Mathematics of Climate, 13 October 2010

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Includes material from talk by S. Yoden
Normal mode calculations by M. Wlasak and K. Ngan
This presentation covers the following areas

• Background
• Understanding observed behaviour using limit equations
• Solving limit equations
• Validating operational algorithms using limit solutions
Background
Governing equations

On all relevant scales, the atmosphere is governed by the compressible Navier-Stokes equations, the laws of thermodynamics, phase changes and source terms.

The solutions of these equations are very complicated, reflecting the complex nature of observed flows.

The accurate solution of these equations would require computers $10^{30}$ times faster than now available.

We can predict the weather quite well, can we do better?
Direct study of the Navier-Stokes equations focusses on viscous behaviour, and gives understanding only of turbulent microscales.

Understanding on any larger scale obtained by approximations to the equations valid in particular asymptotic limits.

Essential that these limit equations can be solved for sufficiently large times to be useful, and that they represent a physically consistent asymptotic regime.
Method

Choose appropriate small parameters.

Examples are Rossby number, Froude number, Mach number, Reynolds number, aspect ratio, ratio of horizontal scale to earth’s radius.

Choose appropriate limit, which may require several parameters to tend to zero in a prescribed ratio.

Correct asymptotic behaviour requires the smallest parameter to be taken to zero before the others.

Increased modelling capability means a wider selection of regimes is now relevant.
Limiting behaviour of global shallow water equations
Hayashi, Nishizawa, Takehiro, Yamada, Ishioka, and Yoden (2007)

Rossby waves and jets in a two-dimensional decaying turbulence on a rotating sphere

Experimental parameters

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>( Fr )</th>
<th>( k_\beta (\phi = 0) )</th>
<th>( L_{Dp} )</th>
<th>( L_{De} )</th>
<th>( \phi )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>( 1/\sqrt{10} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>–</td>
</tr>
<tr>
<td>25</td>
<td>( 1/\sqrt{10} )</td>
<td>5.9</td>
<td>0.063</td>
<td>0.25</td>
<td>0.37 (21°)</td>
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<tr>
<td>50</td>
<td>( 1/\sqrt{10} )</td>
<td>8.4</td>
<td>0.032</td>
<td>0.18</td>
<td>0.26 (15°)</td>
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<tr>
<td>100</td>
<td>( 1/\sqrt{10} )</td>
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<td>0.016</td>
<td>0.13</td>
<td>0.19 (11°)</td>
</tr>
<tr>
<td>400</td>
<td>( 1/\sqrt{10} )</td>
<td>24</td>
<td>0.004</td>
<td>0.063</td>
<td>0.09 (5°)</td>
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<tr>
<td>400</td>
<td>( 1/\sqrt{100} )</td>
<td>24</td>
<td>0.013</td>
<td>0.11</td>
<td>0.30 (17°)</td>
</tr>
<tr>
<td>400</td>
<td>( 1/\sqrt{1000} )</td>
<td>24</td>
<td>0.04</td>
<td>0.2</td>
<td>0.86 (49°)</td>
</tr>
<tr>
<td>400</td>
<td>( 1/\sqrt{10000} )</td>
<td>24</td>
<td>0.13</td>
<td>0.35</td>
<td>1.47 (84°)</td>
</tr>
<tr>
<td>400</td>
<td>0</td>
<td>24</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>–</td>
</tr>
<tr>
<td>4000</td>
<td>( 1/\sqrt{10} )</td>
<td>75</td>
<td>0.0004</td>
<td>0.02</td>
<td>0.03 (1.7°)</td>
</tr>
</tbody>
</table>

important numbers for dynamical understanding:

- Rhines wavenumber: \( k_\beta = \sqrt{(2\Omega \cos \Phi / <u>)} \)
- radius of deformation: \( L_D = 1 / (2 \text{Fr} \Omega \sin \Phi) \)
- a key latitude: \( \Phi@ \) where \( L_D = 1/k_\beta \)
vorticity at $t = 5$ for five $Fr = 1/\sqrt{10} \sim 0$ with $\Omega=400$

green line: $\Phi@$ where $L_D = 1/k_\beta$

it gives an estimate of latitudinal extent of zonally elongated structure in the lower latitudes

coherent vortices appear in the higher latitudes

Jovian atmosphere: $\Omega \sim 300$, $Fr \sim 0.025$

Earth’s atmosphere: $Fr \sim 0.035$

Internal mode of Earth’s ocean: $Fr \sim 0.025$
Limit equations

Small parameters $Ro$ and $Fr$. $L_d/L = Ro/Fr$. Define $\varepsilon = (Ro^{-1} + Fr^{-1})^{-1}$. Consider only cases where $\varepsilon$ small.

2d incompressible Euler accurate to $O(Fr^2)$, but has $O(1)$ errors for $Ro$ small, $Fr = O(1)$. Error measured in vorticity.

QG accurate to $O(Fr)$ if limit is $Fr = Ro$. Errors are $O(1)$ if $Ro \to 0, Fr = O(1)$ and $O(Fr^2)$ in vorticity if $Fr \to 0, Ro = O(1)$.

SG accurate to $O(Ro(Ro/Fr)^2)$ if $Ro \to 0, Ro/Fr \to 0$ and $O(Fr)$ as $Fr \to 0, Ro = O(1)$. Errors measured in height.

BE accurate to at least $O(\varepsilon^2)$. 
Solvability

Need to solve systems and prove that solution of shallow water stays as close to that of limit equation as the *a priori* estimate suggests.

2d Euler and QG can be solved for large times and *a posteriori* error estimate made. Non-optimal for 2d Euler case.

SG can probably be solved for large times on sphere (certainly on f plane). Non-optimal *a posteriori* error estimate made.

BE probably cannot be solved without regularisation.
Demonstration

Data with typical max wind speed 15ms\(^{-1}\).

Mean depth chosen to give gravity wave speed 65ms\(^{-1}\) to 360ms\(^{-1}\). (will show typical tropospheric evolution best matched using speed 140ms\(^{-1}\).)

Gives \(Ro \sim 0.1\), \(Fr \sim 0.05-0.3\).

Shallow water version of Met Office UM. SG and 2d Euler codes as similar as possible.
Data for test
Differences in depth

UM-SG differences

UM-BVE differences
Differences in winds

UM-SG differences

UM-BVE differences
Comments

SG results show linear convergence in $Ro/Fr$, $Ro/Fr$ not small enough to give quadratic.

SG total winds shows no conclusive result, not covered by theory.

2d Euler shows expected convergence in winds, depth asymptotes to non-zero difference.

SG differences much smaller than 2d Euler.
2d turbulence?

Classical 2d turbulence behaviour requires energy and enstrophy to scale differently. Then any cascade has to be upscale in energy and downscale in enstrophy.

Ratio of potential enstrophy to energy ($k=$wavenumber)

<table>
<thead>
<tr>
<th></th>
<th>SWE</th>
<th>2dEuler</th>
<th>SG</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L \ll L_D$</td>
<td>$k^2$</td>
<td>$k^2$</td>
<td>$k^2+$nonlinear</td>
</tr>
<tr>
<td>$L \gg L_D$</td>
<td>1</td>
<td>$k^2$</td>
<td>1</td>
</tr>
</tbody>
</table>
Consequences

No preferred cascade direction for $L>L_D$. Computations suggest upscale cascade suppressed. 2d Euler misses this effect.

SG distorts downscale cascade of potential enstrophy for $L<L_D$.

Correct representation of $L_D$ essential for blocking climatology of atmospheric models.

Blocking patterns do not have vertical structure of external normal mode.
500hpa chart with block
100hpa chart-smoothed out block
Projection of data onto vertical modes

Standard deviation vs mode number
Observed radius of deformation

These results given a ‘mean’ equivalent depth of 2km, thus an \( L_D \) of about 1400km, gravity wave speed 140ms\(^{-1}\).

Corresponds to wavenumber 4 in mid latitudes.

Thus persistent blocking patterns correspond to lower wavenumbers—as observed.

Wave-trapping in lower stratosphere (Charney-Drazin) reduces effective \( L_D \) for tropospheric motions.

Important to represent this correctly in models.
Solving limit equations
Solution procedure for balanced models

Most such models can be written in terms of transport of PV, together with solution of an elliptic equation for the other variables.

The PV is either the Ertel PV, or an approximation to it with the same accuracy as the balanced system of equations as a whole.
Solution of the transport equation

\[ \frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma U) = 0 \]

\( \sigma \) is the potential vorticity, \( U \) is the velocity.

This equation is well-posed and defines a Lagrangian trajectory if \( U \) is \( L^1_{\text{loc}} \) (Diperna and Lions). Ambrosio extended this to \( U \) BV and divergence-free.
The PV inversion problem must generate a divergence-free BV velocity field $U$ for given bounded PV $\sigma$.

We can then prove that $\sigma$ is conserved and remains bounded, so we can solve the system for large times.

In QG and 2d Euler, the PV inversion problem involves solving a (constant-coefficient) Poisson equation. The smoothing property of this equation ensures that $U$ satisfies the requirements.
Large-scale balanced models

On large scales in the atmosphere the assumptions of nearly constant static stability and Coriolis parameter are not valid.

The SG equations are a generalisation of QG including variable static stability and Coriolis parameter, but still restricted to flows controlled by the earth’s rotation.

The balance equations also include variable coefficients, and are valid for weakly rotating and strongly stratified flows as well.
Controlling the coefficients

In SG and BE, the PV inversion problem is nonlinear and state-dependent.

If the balance assumptions are satisfied, then the equation will be elliptic and solvable.

It is necessary to prove that this remains true during a time evolution given suitable initial data.

It is possible to do this for SG, but unlikely to be possible for BE.

In SG this depends on a variational formulation of the problem.
Solution procedure for SG I

Illustrate with shallow water. The equations are

\[ \dot{u}_g = fJ(u_g - u) \]

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ \frac{\partial h}{\partial t} + \nabla \cdot (hu) = 0 \]

Set

\[ f^2 \dot{X} = -J\dot{u}_g + fu \]

Then

\[ \dot{X} = f^2 u_g \]
If \( X \) is regarded as a coordinate on a dual manifold \( M \), and \( \sigma \) is the mass on \( M \), then

\[
\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{U}) = 0
\]

where

\[
\mathbf{U} = f \frac{1}{2} \mathbf{u}_g
\]

is the velocity on \( M \).

If \( f \) is not constant, \( M \) cannot be the same as physical space.
Variational formulation

Need to calculate $U$ given $\sigma$.

Let $\Phi(t,.)$ be a Lagrangian map on $M$ and $F(t,.)$ the corresponding map on physical space.

Consistently with the governing equation, define related virtual displacements by

$$\frac{1}{f^2} \delta \Phi = -J \delta u_g + f \delta F$$
Variational principle

Require $u_g$ and the map between $\Phi$ and $F$ to minimise the energy subject to a mass $\sigma$ on $M$ corresponding to a mass $h$ in physical space.

Consider a fixed point on $\Phi$. The variational formulation means that we must minimise the energy subject to the condition that

$$0 = -J \delta u_g + f \delta F$$

Standard calculations then show that

$$f u_g = g J \nabla h$$
Conversely, fixing a point on $F$ and minimising the energy leads to the condition

$$f^2 u_g(X) = J \nabla H(X)$$

The condition that the energy is minimised, not just stationary, is that

$$Q = f \begin{pmatrix} f + \frac{\partial v_g}{\partial x} & \frac{\partial v_g}{\partial y} \\ \frac{\partial u_g}{\partial x} & f - \frac{\partial u_g}{\partial y} \end{pmatrix}$$

is positive definite.
Consequences

These arguments show that the velocity $U$ on $M$ is divergence free, the property that $Q$ is positive definite implies that $u_g$ is BV. The properties of the minimising map show that $U$ is BV on $M$.

Thus Ambrosio’s theorem is satisfied and (formally) the equation can be solved.

A rigorous proof appears possible by regarding the energy as a cost in a Monge-Kantorovich problem, and proving the existence of a minimising $u_g$. 
Higher order models

It is likely that higher order balanced models can only be solved with extra regularisation. This becomes part of the model.

Estimating the error in such a model is not possible by direct study of the full equations as we don’t know the spectral structure.

Therefore such methods cannot be used to estimate the behaviour of the full equations, without knowing that behaviour in advance.
Estimating error of higher order models

Truncated balanced model should accurately replicate behaviour of truncated Navier-Stokes model.

Assume truncation within observed $k^{-5/3}$ observed range.

If energy quadratic, $L_2$ error will be $K^{-1/3}$ where $K$ is truncation wavenumber.

Consistent with rate of improvement of operational models as $K$ increases.
Value of models that can be solved

Solvability of balanced model only breaks down if $Ro=O(1)$ anywhere in the solution.

If solution does not change qualitatively as $Ro$ decreases, then expect $K$ to be proportional to $Ro^{-1}$.

Accuracy of balanced model then $O(Ro^{1/3})$

QG result shows that we can get $O(Ro)$. This proves that solution must change qualitatively with decreasing $Ro$, could not deduce this from balanced model analysis.
Validation of frontal solutions
Eady problem

Consider fully compressible equations in a vertical cross-section \((x,z)\) with periodic boundary conditions in \(x\) and rigid upper and lower boundaries.

The SG limit is obtained by neglecting \((Du/Dt,Dw/Dt)\).

Existence of a solution to the SG equations in this case has been proved.

Scale analysis shows that SG is accurate to \(O(Ro^2)\) in this geometry.
Limit equations

These equations can be written in Lagrangian form as

\[
\frac{DX}{Dt} = C \left( z - \frac{H}{2} \right)
\]

\[
\frac{DZ}{Dt} = C \left( X - x \right)
\]

\[X + (0, g) + Z \nabla \Pi = 0\]

\[F \# \rho (0, \cdot) = \rho (t, \cdot)\]

\(F\) is the Lagrangian map, \(\Pi\) the Exner pressure, \(X = v + fx\), \(Z = \theta\), \(C\) represents a basic state potential temperature gradient in \(y\).
Properties

The theory shows that weak solutions of the Lagrangian equations exist.

Discontinuities can form. In general weak solutions of the Eulerian equations do not exist.

Suggests that numerical methods respecting this limit have to discretise the Lagrangian form of the equations.

Brenier has proved (in the incompressible case) that solutions of the Euler equations converge to those of SG at a rate $Ro^{\frac{1}{2}}$, this is well short of the optimal rate.
Computational test

2d cross-section. Initial data is fastest growing Eady normal mode.

Equations produce identical balanced (SG) solution if rescaled $L \rightarrow \beta L, U \rightarrow \beta U, f \rightarrow \beta f$ (f is Coriolis parameter)

This rescaling replaces $Ro$ by $\beta Ro$.

Use Met Office UM (fully compressible) scheme with semi-lagrangian advection-appropriate for discontinuous Lagrangian solutions.

Check limiting behaviour as $Ro \rightarrow 0$ by using UM with $Ro=0.031$ as reference.

Check ratio of unbalanced along-front wind to total wind.
Computing the limit

The UM uses a staggered grid suitable for representing balance in the vertical but not the horizontal.

The UM grid is optimised for allowing balance to be achieved.

The UM uses semi-Lagrangian advection-consistent with Lagrangian solutions.

Conservative semi-Lagrangian advection as under trial in new version of the UM should be used for full consistency with the limit solutions.
Convergence test-wind along front

Standard

Monotone
Convergence test-potential temperature

Standard

Monotone
Convergence test-imbalance

Standard

Monotone
Comments

Second order convergence in $Ro$ shown for winds, temperatures while solutions are smooth. Agrees with a priori estimate.

Value of differences from limit reduced by using quasi-monotone advection.

Imbalance tends to zero at second order rate. Shows that limit does satisfy SG equations.
Results for discontinuous case show approximately first order convergence rate, increased when quasi-monotone advection used.

Confirms that SG is a valid limit of Euler even when it is discontinuous.

Justifies argument that this limit should be respected in operational models.
Applications to the real system
Summary

Demonstrated use of reduced models to explain behaviour of more general equations in different regimes.

Illustrated what is needed to solve reduced models valid on large scales in the atmosphere.

Used the theory of a reduced model to validate numerical methods used in an operational model.

Used the computations to suggest what degree of approximation should exist beyond what has been proved to date.