Derivation of a Local Form of the Maximum Entropy Production Principle

Mathematical and Statistical Approaches to Climate Modelling and Prediction,
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Will be co-host (with Roddy Dewar) of a “maximum entropy methods workshop”, Canberra, Australia, in Sept. 2011

Meeting on maximum entropy and entropy production-related themes:
- theory
- planetary system dynamics (atmosphere, oceans, mantle, biosphere, etc)
- fluid turbulence + convective heat transfer systems
- cosmology / astrophysics / astrobiology
- biological / ecological systems
- solid mechanics (elastic / plastic / viscous deformation + failure)
Contents

1. Concepts
   - analysis of probabilistic systems: Jaynes’ MaxEnt; Boltzmann’s MaxProb
   - types of systems
   - the maximum entropy production (MaxEP) principle

2. Operation of MaxEnt
   - MaxEnt algorithm + implications
   - equilibrium systems → equilibrium position
   - role of Planck potential (= free energy / temperature)

3. Analysis of local steady-state system
   - apply MaxEnt → steady-state position
   - role of potential $\phi_{st}$

4. Analysis of global steady-state system (Dewar)
   - discussion + critique
Concepts
Analysis of Probabilistic Systems

Consider any system of discrete entities
e.g. atoms / molecules / ions, quantum particles,
biological organisms, transportation units,
economic agents, social actors, humans

- maximise entropy of a system, subject to constraints
→ inferred (“least informative”) description of system

Boltzmann (1877), Planck (1901) MaxProb principle:
- maximise entropy of a system, subject to constraints → most
  probable position of system
→ general principle of inference (any probabilistic system)
“Measures of Central Tendency”

Continuous $x \in \mathbb{R}$

$P(x) = \text{probability density function (pdf)}$

Discrete $m \in \mathbb{Z}$

$P_m = \text{probability distribution}$

$W_m = \text{statistical weight (frequency)}$
Definitions

Relative entropy of $\alpha$th type of system:

$$S_j^{\alpha} = -\sum_{i=1}^{s} p_i^{\alpha} \ln \frac{p_i^{\alpha}}{q_i^{\alpha}}$$

(will assume this entropy function applies)

where $p_i^{\alpha}$ = probability of $i$th state or category
$q_i^{\alpha}$ = prior (source) probability of $i$th category, in absence of any constraints (e.g. degeneracy)
$s$ = number of categories

Constraints:

$$\sum_{i=1}^{s} p_i^{\alpha} = 1$$

(normalisation constraint)

$$\sum_{i=1}^{s} p_i^{\alpha} f_{ri}^{\alpha} = \langle f_r^{\alpha} \rangle, \quad r = 1, \ldots, R$$

(moment constraints)

where $f_{ri}^{\alpha}$ = value of $i$th category of $r$th constraint
$$\langle f_r^{\alpha} \rangle$$ = expectation value of $r$th constraint
e.g. 1: Thermodynamic Systems

\[ p_i = \text{joint prob. that entity has particular contents } f_{ri} \]

\[ \text{e.g. energy level, volume level, nos. of particles} \]

Constraints = *mean contents* of system \( \langle f_r \rangle \)

(a) **Isolated system**
- fixed contents
- entities = molecules
  (microcanonical ensemble)

(b) **Open (diffusive) system**
- system surrounded by bath
- entities = entire systems
  (canonical, grand canonical ensembles)

Apply MaxEnt \( \rightarrow \) *equilibrium position* of system
e.g. 2: Steady-State Flow Systems

(a) **Global** (Dewar, 2003, 2005)
Analyse entire control volume
\[ p_i = \text{joint prob. that system has instantaneous fluxes } j_{ri}(x) \text{ around boundary} \]
Constraints = *mean boundary fluxes* \( \langle j_r \rangle(x) \)
e.g. heat, momentum, nos. of particles

(b) **Local** (Niven, 2009, 2010)
Analyse local infinitesimal element
\[ p_i = \text{joint prob. of instantaneous fluxes } j_{ri} \text{ through element} \]
Constraints = *mean fluxes* \( \langle j_r \rangle \text{ through element} \)

Apply MaxEnt (different entropy) \( \rightarrow \) *steady-state position of system*
MaxEnt / MaxProb for Systems Analysis

Contention:
MaxEnt / MaxProb method $\rightarrow$ dramatic simplifications to systems analysis
- thermodynamic context $\Rightarrow$ generic context
- old + new phenomena
- model closure (?)

Caveats:
- probabilistic, not deterministic (this is inference !)
- discard all unnecessary information (e.g. details of dynamics)
Maximum Entropy Production (MaxEP) Principle

“A dynamic flow-controlled system seeks a steady state at which there is a maximum rate of production of thermodynamic entropy”

e.g. earth climate system
Application of MaxEP

Earth Climate System
Paltridge (1975, 1978):
- 10-box (latitudinal) model of atmosphere-ocean system
- atmosphere, oceanic + overall energy balances → many solutions
- choose heat fluxes which maximise entropy production:

![Diagram](image)
Application of MaxEP

Can also apply to:
- convective heat transfer systems (Rayleigh-Bénard cell)
- turbulent fluid flow systems
- biological (chemically degrading) systems

So where does MaxEP come from?

Note: MaxEP is fundamentally different to Prigogine’s MinEP principle:
- MaxEP: choice of observed steady state from set of possible steady states
- Prigogine MinEP: selection of steady state from set of non-steady state solutions (trivial)
Operation of Jaynes’ MaxEnt
Consider a type of probabilistic system (of any kind):

Maximise \( \mathcal{S}_\alpha = -\sum_{i=1}^{s} p_i^\alpha \ln \frac{p_i^\alpha}{q_i^\alpha} \) relative entropy

subject to \( \sum_{i=1}^{s} p_i^\alpha = 1 \) normalisation constraint

\( \sum_{i=1}^{s} p_i^\alpha f_r = \langle f_r^\alpha \rangle, \quad r = 1, \ldots, R \) moment constraints

Write Lagrangian, extremise \( p_i^\alpha \rightarrow \) most probable realisation:

\( p_i^{\alpha^*} = \frac{1}{Z^\alpha} q_i^\alpha e^{\sum_{r=1}^{R} \lambda_r^\alpha f_r^\alpha} \) Boltzmann distribution

where \( \lambda_r^\alpha = \) Lagrangian multipliers

\( Z^\alpha = \sum_{i=1}^{s} q_i^\alpha e^{\sum_{r=1}^{R} \lambda_r^\alpha f_r^\alpha} = \) partition function

\( \lambda_0^\alpha = \ln Z^\alpha = \) Massieu function
Jaynes’ Relations

Maximum entropy:
\[ \mathcal{H}_\alpha^* = \ln Z^\alpha + \sum_{r=1}^{R} \lambda_r^\alpha \langle f_r^\alpha \rangle \]

Minimum potential:
\[ \phi^\alpha = -\ln Z^\alpha = -\mathcal{H}_\alpha^* + \sum_{r=1}^{R} \lambda_r^\alpha \langle f_r^\alpha \rangle \]

Derivatives:
\[ \frac{\partial \mathcal{H}_\alpha^*}{\partial \langle f_r^\alpha \rangle} = \lambda_r^\alpha \]
\[ \frac{\partial \phi^\alpha}{\partial \lambda_r^\alpha} = \langle f_r^\alpha \rangle \]
\[ \frac{\partial^2 \mathcal{H}_\alpha^*}{\partial \langle f_m^\alpha \rangle \partial \langle f_r^\alpha \rangle} - \frac{\partial^2 \phi^\alpha}{\partial \lambda_m^\alpha \partial \lambda_r^\alpha} = g_{mr} \in \mathbf{g} \]
\[ \frac{\partial \langle f_r^\alpha \rangle}{\partial \lambda_m^\alpha} = \frac{\partial \langle f_m^\alpha \rangle}{\partial \lambda_r^\alpha} \]
\[ \frac{\partial \langle f_r^\alpha \rangle}{\partial \lambda_m^\alpha} = \frac{\partial \langle f_m^\alpha \rangle}{\partial \lambda_r^\alpha} \]
\[ \frac{\partial^2 \phi^\alpha}{\partial \lambda_m^\alpha \partial \lambda_r^\alpha} = -\text{cov}(f_m^\alpha,f_r^\alpha) = \gamma_{mr} \in \gamma \]

Legendre transf.:
\[ \gamma = g^{-1} \quad \text{for} \quad \phi^\alpha(\lambda_1^\alpha,\lambda_2^\alpha,...) \Leftrightarrow \mathcal{H}_\alpha^*(\langle f_1^\alpha \rangle,\langle f_2^\alpha \rangle,...) \]
Interpretation 1

Variation of constraint (Jaynes, 1957, 1963):

\[ d\langle f_r^\alpha \rangle = d\left(\sum_{i=1}^{S} p_i^\alpha f_{ri}^\alpha\right) = \sum_{i=1}^{S} p_i^\alpha df_{ri}^\alpha + \sum_{i=1}^{S} dp_i^\alpha f_{ri}^\alpha \]

\[ \delta W_r^\alpha = \text{gen. work} \quad \delta Q_r^\alpha = \text{gen. heat} \]

Generic Clausius equality

\[ d\phi^\alpha = \sum_{r=1}^{R} \lambda_r^\alpha \delta Q_r^\alpha \]

Generic potential

\[ d\phi^\alpha = \sum_{r=1}^{R} \lambda_r^\alpha \delta W_r^\alpha + \sum_{r=1}^{R} d\lambda_r^\alpha \langle f_r^\alpha \rangle \]

If constant multipliers \{\lambda_r^\alpha\}, \phi^\alpha = \text{weighted generalised } \alpha\text{th type of work}

Minimum \phi^\alpha \rightarrow \text{position of minimum available } \alpha\text{th type of work}
Interpretation 2

Most systems: **must** consider changes in entropy within + outside system:

\[ d\mathcal{S}_\text{univ}^\alpha = d\mathcal{S}_\alpha^* + d\mathcal{S}_\text{ext}^\alpha \geq 0 \]

**Generic second law**

But any external change \(\leftrightarrow\) changes in constraints and/or multipliers

\[ d\mathcal{S}_\text{ext}^\alpha = -d \sum_{r=1}^{R} \lambda_r^\alpha \langle f_r^\alpha \rangle = \frac{d\sigma^\alpha}{\kappa} \]

**Generic entropy production**

\[ \therefore \quad d\phi^\alpha = -d\mathcal{S}_\alpha^* + d \sum_{r=1}^{R} \lambda_r^\alpha \langle f_r^\alpha \rangle = -d\mathcal{S}_\text{univ}^\alpha = -d\mathcal{S}_\alpha^* - \frac{d\sigma^\alpha}{\kappa} \leq 0 \]

\[ \therefore \quad \phi^\alpha = \text{generic, dimensionless, free energy concept} ! \]

Minimum \(\phi^\alpha \rightarrow\) position of maximum \(\alpha\)th entropy of universe

If constraints can vary, **must** minimise \(\phi^\alpha\), **not** maximise \(\mathcal{S}_\alpha^*\)
**Thermodynamic System**

e.g. \[ \sum_{i=1}^{s} p_i = 1 \] natural constraint

\[ \sum_{i=1}^{s} p_i h_i = \langle H \rangle \] mean enthalpy

Apply MaxEnt (identify \( \lambda_H = 1/kT \) at const. \( P \))

\[ p_i^* = \frac{1}{Z} e^{-\lambda_H h_i} = \frac{1}{Z} e^{-h_i/kT} \]

\[ S^* = k \ln Z + \frac{\langle H \rangle}{T} \]

\[ k \ d\phi_{eq} = d \left( \frac{G}{T} \right) = -dS^* + d \frac{\langle H \rangle}{T} \leq 0 \]

Min \( \phi_{eq} \) \( \rightarrow \) interplay between internal & external changes in entropy:

<table>
<thead>
<tr>
<th>Max ( S^* )</th>
<th>Max ( \sigma_{eq} )</th>
<th>Universal</th>
</tr>
</thead>
<tbody>
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<td>Max ( S^* )</td>
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\( i=s \)

\( i=3 \) Energy level

\( i=2 \)

\( i=1 \) Particle

= \(-d\sigma_{eq} = \) \(-\) increment of entropy produced
### Other Ensembles

<table>
<thead>
<tr>
<th>Quantity Constraints</th>
<th>Multipliers</th>
<th>Potential $k\phi_{eq}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural, $\langle U \rangle$</td>
<td>$\lambda_U$</td>
<td>$F = -k\ln Z_1 = -S^* + \frac{\langle U \rangle}{T}$</td>
</tr>
<tr>
<td>Natural, $\langle H \rangle$</td>
<td>$\lambda_H$</td>
<td>$G = -k\ln Z_2 = -S^* + \frac{\langle H \rangle}{T}$</td>
</tr>
<tr>
<td>Natural, $\langle H \rangle, {\langle N_c \rangle}$</td>
<td>$\lambda_H, {\lambda_c}$</td>
<td>$J = -k\ln Z_3 = -S^* + \frac{\langle H \rangle}{T} - \sum_{c=1}^{C} \frac{\mu_c \langle N_c \rangle}{T}$</td>
</tr>
<tr>
<td>Natural, $\langle u \rangle, \langle \bar{e} \rangle$ (per unit mass quantities)</td>
<td>$\lambda_U, \lambda_{\bar{e}}$</td>
<td>$\gamma = -k\ln Z_4 = -s^* + \frac{\langle u \rangle}{T} + \frac{\bar{\pi} : \langle \bar{e} \rangle}{\rho T}$</td>
</tr>
</tbody>
</table>

**Identify**  
$\lambda_U = \left. \frac{1}{kT} \right|_V$, $\lambda_H = \left. \frac{1}{kT} \right|_P$, $\lambda_c = -\frac{\mu_c}{kT}$, $\lambda_{\bar{e}} = -\frac{\bar{\pi}}{k\rho T}$

$F$ = Helmholtz free energy, $G$ = Gibbs free energy  
$J, \gamma$ given by Gibbs (1875-78) (modified $J \rightarrow$ exergy concept)
Summary

• Jaynes’ MaxEnt
  → all of existing equilibrium thermodynamics
  → new ensembles

• Minimum $\phi_{eq}$ → infer equilibrium position
  - Massieu (1869), Planck (1932):
    - minimise $k\phi_{eq}$ → “Planck potential” in entropy units
  - Gibbs (1875-1878):
    - assume $T$ constant
    - minimise $kT\phi_{eq}$ → potential expressed in energy units
Analysis of Local Steady-State Flow
Local Analysis

1. Discretise control volume
   → consider infinitesimal (or small) boundary element, with mean local fluxes:
   \[ \langle j_Q \rangle = \text{mean heat flux vector} \]
   \[ \langle j_c \rangle = \text{mean particle flux vector, species } c \]
   \[ \langle \tau \rangle = \text{mean momentum flux tensor} \]
   \[ = \text{mean viscous stress tensor } \langle \tau \rangle - \langle P \rangle \delta \]
   (will omit chemical reactions here)

2. Adopt “local equilibrium” assumption
   → local \( T, P, \{ \mu_c \}, \rho, \text{etc}, \) within volume elements on either side of boundary
Control Volume Analysis
(de Groot & Mazur, 1984; Bird et al. 2006)

Entropy production across boundary element is:

\[
\hat{\sigma} = \langle \mathbf{j}_Q \rangle \cdot \nabla \left( \frac{1}{T} \right) - \sum_c \langle \mathbf{j}_c \rangle \cdot \nabla \left( \frac{\mu_c}{M_c T} - \frac{\mathbf{g}_c}{T} \right) - \langle \vec{\tau} \rangle : \nabla \left( \frac{\mathbf{v}}{T} \right)^\top
\]

- heat diffusion
- chemical + field diffusion
- momentum diffusion

where \( M_c \) = molar mass of \( c \) \( \mathbf{g}_c \) = body force vector on \( c \) \( \mathbf{v} \) = fluid velocity

**Problem:** in dissipative systems, \( \hat{\sigma} \) is indeterminate
(entropy not conserved; require knowledge of fluxes AND gradients)
MaxEnt Analysis
(Niven, PRE 80(2) (2009) 021113; Phil Trans B 365: (2010) 1323-1331)

Consider infinitesimal boundary element with
- mean fluxes / rates $\langle j_Q \rangle, \langle j_c \rangle, \langle \bar{\tau} \rangle$
- instantaneous fluxes / rates $j_{Q,I}, \{j_{Nc}\}, \bar{\tau}_J$

Maximise flux entropy $\mathcal{H}_{st} \rightarrow$ most probable steady-state description of element:

$$p_i^* = \frac{1}{Z} q_i \exp \left( -j_{Q,I} \cdot \zeta_Q - \sum_c j_{Nc} \cdot \zeta_c - \bar{\tau}_J : \zeta_\tau \right)$$

$$\mathcal{H}_{st}^* = \ln Z + \langle j_Q \rangle \cdot \zeta_Q + \sum_c \langle j_c \rangle \cdot \zeta_c + \langle \bar{\tau} \rangle : \zeta_\tau$$

where $\zeta_r = \text{Lagrangian multipliers}$

Since constraints are linearly independent, identify multipliers $\propto$ gradients, hence

$$\mathcal{H}_{st}^* = \ln Z - \frac{\hat{\sigma}}{\mathcal{K}}$$

where $\mathcal{K} = \text{constant (units of J K}^{-1} \text{ m}^{-2} \text{ s}^{-1})$
Hence must minimise potential:

\[ \phi_{st} = -\ln Z = -\mathcal{S}_{st}^* - \frac{\hat{\sigma}}{\mathcal{K}} \]

Again have interplay between changes of entropy (here the flux entropy \( \mathcal{S}_{st}^* \)) within and outside system:

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Only two of these are consistent with a local MaxEP principle!
Conclude:

1. Can derive a local conditional MaxEP principle, based on MaxEnt
2. MaxEP is not universal - it is only one possibility!
   (c.f. minimum enthalpy principle in thermodynamics)

\[ d\hat{\phi}_{st} = -d\xi^*_{st} - \frac{d\hat{\sigma}}{\mathcal{K}} \leq 0 \]

\[ d\hat{\phi}_{eq} = d \frac{G}{kT} = -d\frac{S^*}{k} + d\frac{\langle H \rangle}{kT} \leq 0 \]
Analysis of
Global Steady-State Flow
Global Analysis

(Dewar, 2003, 2005)

Analyse entire control volume

Constraints = mean boundary fluxes $\langle j_r \rangle(x)$
  e.g. heat, momentum, nos. of particles

Derivation cast in terms of a path entropy

$$\tilde{S}_\Gamma = -\sum_{\Gamma} p_{\Gamma} \ln \frac{p_{\Gamma}}{q_{\Gamma}}$$

where $p_{\Gamma}$ = joint prob. that system has particular fluxes at boundaries

Advantage

1. Analysis conducted at (global) scale of interest
   → omit details of local subsystems
**Objections**

1. (Language): Does not examine history-dependent effects
   ∴ Γ are not “paths”, but the global flux states of the system

2. (Reasoning): Have not yet “proven” global MaxEP
   - argument depends on a complicated double optimisation, which encounters difficulties
   - should in fact follow above Jaynes analysis *in entirety*
     → obtain potential function to be minimised

\[
    d\phi_\Gamma = -d\mathcal{H}^*_\Gamma - \frac{d\mathcal{S}_\Gamma}{\mathcal{K}} \leq 0
\]

→ again get interplay between (global) \(d\mathcal{H}^*_\Gamma\) and \(d\mathcal{S}_\Gamma\)
→ MaxEP important but not universal
3. (Scale): How does a system “know” that it is a system?
   - can we subdivide it?
   - can a system contain subsystems in which $\text{EP}<0$, compensated by others in which $\text{EP}>0$?
     - NO! - if so, could draw a boundary around EP<0 system, which would continuously violate 2nd law of thermodynamics
   - hence any MaxEP (or minimum $\phi_{st}$) principle must apply at all scales (at which we can reasonably define macroscopic variables)
Unfinished Business

1. Possible to demonstrate (have not proven):

\[ \dot{\sigma}_{\text{global}} = \sum_{\text{internal boundaries } \Omega} A_{\Omega} \dot{\sigma}_{\Omega} \]

provided one considers all local boundaries \( \Omega \) between sub-volumes (each at local equilibrium)

2. Need to prove (or disprove):

\[ \text{Max } \dot{\sigma}_{\text{global}} = \sum_{\text{internal boundaries } \Omega} A_{\Omega} \text{Max } \dot{\sigma}_{\Omega} \]

where the local maxima are constrained by flows between internal elements

3. What does a mean mean?
   - how variable can the constraints \( \langle \mathbf{j}_r \rangle \) be?
   - in fact “steady state” is anything but steady!
Conclusions
Conclusions

1. Concepts
   - analysis of probabilistic systems: MaxEnt; MaxProb
   - the maximum entropy production (MaxEP) principle

2. Operation of MaxEnt
   - algorithm + implications → potential function $\phi$
   - equilibrium systems → equilibrium position; Planck potential

3. Analysis of local steady-state system
   - apply MaxEnt → local steady-state position
   - minimum potential $\phi_{st}$ → conditional MaxEP principle

4. Analysis of global steady-state system (Dewar)
   - discussion + critique
   - need to consider potential $\phi_\Gamma$

Further work to reconcile global + local analyses
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Thank you!
Meaning of Flux Entropy?
(Niven, Phil Trans B 365: (2010) 1323-1331)

- distinct from thermodynamic entropy $S^*$
- expresses spread of $p^*_i$ over instantaneous fluxes
- unlike equilibrium systems, have $i \in \mathbb{Z}$

For univariate flux levels $i$

Increasing $\mathcal{H}^*_i$ \\

Hence $\mathcal{H}^*_i \uparrow \iff$ more fluctuating flow, with instantaneous flow reversals
(turbulent flow; ecological populations; economic systems)
**Laws of Thermostatics**  
(Equilibrium system)

0th

\[ \lambda_{r1} + \lambda_{r2} \rightarrow \lambda_{r} \]

1st

\[ d\langle f_r \rangle = \delta W_r + \delta Q_r \]

2nd

\[ dS^* = k \sum_{r=1}^{R} \lambda_r \delta Q_r \]
\[ kd\phi_{eq} = d \frac{G}{T} = -dS^* + d \left( \frac{\langle U \rangle}{T} + \frac{P\langle V \rangle}{T} \right) \]

3rd

\[ S^* = \ln Z + \sum_{r=1}^{R} \lambda_r \langle f_r \rangle \xrightarrow{p_1 \rightarrow 1} 0 \]

**Laws of Thermodynamics**  
(Steady state system)

0th

\[ \zeta_{r1} + \zeta_{r2} \rightarrow \zeta_{r} \]

1st

\[ d\langle j_r \rangle = \delta w_r + \delta q_r \]

2nd

\[ d\hat{S}_{st} = \sum_{r=1}^{R} \zeta_r \cdot \delta q_r \]
\[ d\phi_{st} = -d\hat{S}_{st} - \frac{1}{K} d\hat{\delta} \]

3rd

\[ \hat{S}_{st} = \ln Z + \sum_{r=1}^{R} \zeta_r \cdot \langle j_r \rangle \xrightarrow{p_0 \rightarrow 1} 0 \]
**Expansion**

Expand flux \( \langle j_{rij} \rangle \) for \( i, j \in \{x, y, z\} \) about \( \{\zeta_r = 0\} \):

\[
\langle j_{rij} \rangle = \sum_{mkl} \left. \frac{\partial \langle j_{ij} \rangle}{\partial \zeta_{mkl}} \right|_{\{\zeta_r = 0\}} \zeta_{mkl} + \frac{1}{2!} \sum_{mkl} \sum_{n\varphi\theta} \left. \frac{\partial^2 \langle j_{ij} \rangle}{\partial \zeta_{mkl} \partial \zeta_{n\varphi\theta}} \right|_{\{\zeta_r = 0\}} \zeta_{mkl} \zeta_{n\varphi\theta} + \ldots
\]

---

Onsager linear transport regime

But always have Jaynes reciprocal relation (local):

\[
\frac{\partial \langle j_{rij} \rangle}{\partial \zeta_{mkl}} = \frac{\partial \langle j_{mkl} \rangle}{\partial \zeta_{rij}}
\]

(near and far from equilibrium)
**Generalised Riemannian Geometry**

- Jaynes’ analysis

  \[ \frac{\partial^2 \phi}{\partial \lambda_m \partial \lambda_r} = \gamma_{mr} \] as Riemannian metric on manifold of stationary positions

- arc length:

  \[ L_\phi = \int_{0}^{\xi_{\text{max}}} \sqrt{\sum_{m,r=1}^{R} \gamma_{mr} \frac{\partial \lambda_m}{\partial \xi} \frac{\partial \lambda_r}{\partial \xi}} \, d\xi \]

- distance:

  \[ J_\phi = \int_{0}^{\xi_{\text{max}}} \frac{1}{2} \sum_{m,r=1}^{R} \gamma_{mr} \frac{\partial \lambda_m}{\partial \xi} \frac{\partial \lambda_r}{\partial \xi} \, d\xi \]

Can show:

\[ J_\phi \geq \frac{L_\phi^2}{2\xi_{\text{max}}} \]

Generalised “Least Action Bound”

→ minimum cost (in units of \( \phi \) = units of \( J_\phi \)) to move system from one stationary position to another, at specified rates
Applications

(a) **Equilibrium systems**  
(Salamon, Andresen, Berry, Nulton *et al.*, 1980s-present)
- least action bound $\rightarrow \textit{min. entropy cost}$ of transition between equilibrium positions
- applied to cycle $\rightarrow \textit{min. entropy production principle}$

(b) **Flow systems**  
(Niven & Andresen, 2009)
- least action bound $\rightarrow \textit{min. entropy production principle}$ for transition between steady states of a flow system