Applications of Discrete Harmonic Analysis, Probabilistic Method and Linear Algebra in Fixed-Parameter Tractability and Kernelization

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Outline

1. Introduction
2. Using Probabilistic Method and Harmonic Analysis for $k$-MaxLin-AA
3. Using Linear Algebra for $k$-MaxLin-AA
4. More Applications of Linear Algebra, Probabilistic Method and Hypercontrative Inequality
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Kernelization

- A parameterized problem $\Pi$: a set of pairs $(x, k)$ where $x$ is the main part and $k$ (usually an integer) is the parameter; $x$ is an instance (usually $k \ll |x|$).

- Example 1 ($k$-VertexCover): Given a graph $G = (V, E)$, decide if $\exists U \subseteq V$ s.t. every edge has a vertex in $U$ and $|U| \leq k$. Parameter: $k$. 

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Fixed-Parameter Tractability and Kernelization
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- Example 1 \((k\text{-VertexCover})\): Given a graph \(G = (V, E)\), decide if \(\exists U \subseteq V\) s.t. every edge has a vertex in \(U\) and \(|U| \leq k\). Parameter: \(k\).

- Example 2 \((k\text{-IndependentSet})\): Given a graph \(G = (V, E)\), decide if \(\exists U \subseteq V\) s.t. no edge has both vertices in \(U\) and \(|U| \geq k\). Parameter: \(k\).
Kernelization

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- Example 1 (k-VertexCover): Given a graph $G = (V, E)$, decide if $\exists U \subseteq V$ s.t. every edge has a vertex in $U$ and $|U| \leq k$. Parameter: $k$.
- Example 2 (k-IndependentSet): Given a graph $G = (V, E)$, decide if $\exists U \subseteq V$ s.t. no edge has both vertices in $U$ and $|U| \geq k$. Parameter: $k$.
- Example 3 (k-MaxLin-AA): We are given a system of linear equations over $\mathbb{F}_2$: $\sum_{i \in I_j} y_i = b_j$, $j \in [m]; I_j \subseteq [n]$, and each equation $j$ has a positive integral weight $w_j$. Decide if there is an assignment of values to $y_i$’s s.t. the total weight of satisfied equations is at least $k + \frac{1}{2} \sum_{j=1}^{m} w_j$. 

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Fixed-Parameter Tractability and Kernelization
A parameterized problem is fixed-parameter tractable (fpt) if it can be solved in time $f(k)|I|^{O(1)}$.

Example: runtime $T_1 = 2^k|x|$ is often much smaller than $T_2 = |x|^k$. For $|x| = 1000$, $k = 10$, $T_1 < 1s$, $T_2$ is infeasible.

$k$-VertexCover is fpt.

$k$-IndependentSet is W[1]-hard and thus highly unlikely to be fpt.

Mahajan, Raman and Sikdar (IWPEC’06 & JCSS 2009): What is the complexity of $k$-MaxLin-AA?
A kernelization of $\Pi$: a poly-time algorithm that maps an instance $(x, k) \in \Pi$ to an instance $(x', k') \in \Pi$ (the kernel) such that

- $(x, k)$ is $\text{YES}$ iff $(x', k')$ is $\text{YES}$
- $k' \leq h(k)$ and $|x'| \leq g(k)$ for some functions $h$ and $g$.

- $g(k)$ is the size of the kernel.
- $k$-VertexCover has a kernel with $\leq 2k$ vertices and $k^2$ edges. Size $m + n = k^2 + 2k$ (quadratic).
A decidable parameterized problem is fixed-parameter tractable iff it admits a kernelization. So, $k$-IndependentSet has no kernel (unless FPT = $W[1]$).

Wanted: low degree polynomial-size kernels (for preprocessing).

Many fpt problems do not have polynomial-size kernels (unless coNP $\subseteq$ NP/poly).

$k$-VertexCover has a poly-size kernel.

$k$-MaxLin-AA has size $mn$. Mahajan, Raman and Sikdar (IWPEC’06 & JCSS 2009): Is there any kernel for $k$-MaxLin-AA?
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MaxLin-AA Reduction Rules

- Rule 1: Reduce the system such that no two equations have the same set of variables.

- Rule 2: Let \( a_{ij} = 1 \) if \( y_i \) is in equation \( j \) and \( a_{ij} = 0 \), otherwise. Reduce the system such that \( \text{rank} A = n \), the number of variables, over \( \mathbb{F}_2 \). Thus, \( n \leq m \).

- After applying Rule 1 as long as possible and then Rule 2 as long as possible, we get an irreducible system (no further reductions are possible).
We consider an irreducible system.

- Each equation can be written in the ‘product form’:
  \[ \prod_{i \in I_j} x_i = c_j, \text{ where } x_i = -1 \text{ if } y_i = 1 \text{ and } x_i = 1 \text{ if } y_i = 0, \]
  and \( c_j = (-1)^{b_i} \).

- \( k \)-MaxLin-AA can be written in the ‘function form’: Let
  \[ f(x) = \sum_j d_j \prod_{i \in I_j} x_i, \text{ where } d_j = c_j w_j. \]
  Then the answer to \( k \)-MaxLin-AA is \( \text{YES} \) iff \( \max_{x \in \{-1,1\}^n} f(x) \geq 2k \).
Strictly Above/Below Expectation Method (SABEM): Symmetric Case

- Gutin, Kim, Szeider and Yeo, IWPEC 2009 and JCSS 2011.
- Given a parameterized problem $\Pi$ with parameter $k$.
- Apply some reduction rules.
- Introduce a random variable $X$ s.t. $\mathbb{E}(X) = 0$ and if $\text{Prob}(X \geq k) > 0$ then the answer to $\Pi$ is \textsc{yes}.
- If $X$ is symmetric, then $\text{Prob}[X \geq \sqrt{\mathbb{E}[X^2]}] > 0$.
- If $\sqrt{\mathbb{E}[X^2]} \geq k$ then \textsc{yes}. Otherwise, $\sqrt{\mathbb{E}[X^2]} < k$ and problem specific analysis is required.
SABEM: Asymmetric Case

Lemma (Alon, Gutin, Kim, Szeider, Yeo, SODA 2010)

Let $X$ be a real random variable and suppose that its first, second and forth moments satisfy $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = \sigma^2 > 0$ and $\mathbb{E}[X^4] \leq b(\mathbb{E}[X^2])^2$, respectively. Then $\text{Prob}\left[ X > \frac{\sigma}{2\sqrt{b}} \right] > 0$.

How to check $\mathbb{E}[X^4] \leq b(\mathbb{E}[X^2])^2$?
Hypercontractive Inequality and its ‘Dual’

Let \( f = \sum_{I \subseteq [n]} \hat{f}(I) \prod_{i \in I} x_i \), where \( \hat{f}(I) \) are reals and each \( x_i \in \{-1, 1\} \). Assign value to each \( x_i \) randomly, uniformly and independently from the other variables. Then \( f \) is a random var.

Lemma (Hypercontractive Inequality (HI), Bonami, 1970)

Let \( r = \max \{|I| : \hat{f}(I) \neq 0\} \). Then \( \mathbb{E}[f^4] \leq 9^r \mathbb{E}[f^2]^2 \).

Lemma (‘Dual’ HI, Gutin and Yeo, arXiv June 2011)

Let \( \rho \) is the maximum number of appearances of a number \( i \) in \( I \) for which \( \hat{f}(I) \neq 0 \). Then \( \mathbb{E}[f^4] \leq (2\rho + 1 - \frac{2\rho}{m}) \mathbb{E}[f^2]^2 \), where \( m = |\{I : \hat{f}(I) \neq 0\}| \).

For \( \rho = 1 \) it is tight, e.g., for \( f(x) = 1 + \sum_{i=1}^{n} x_i \). Example: \( f(x) = \prod_{i=1}^{n} x_i \).
Consider \( k \)-MaxLin-AA in the function form:

\[
f(x) = \sum_{j=1}^{m} d_j \prod_{i \in I_j} x_i, \quad n \leq m.
\]

- Suppose that \( \exists \ U \subseteq [n] \) s.t. \( |U \cap I_j| \) is odd for each \( j \).
- Then \( f \) is a symmetric random variable.
- By Parseval’s Identity, \( \mathbb{E}[f^2] = \sum_{j=1}^{m} d_j^2 \geq m \).
- Thus, \( \text{Prob}[ f \geq \sqrt{m} ] > 0 \).
- If \( \sqrt{m} \geq 2k \) then \( \text{YES} \). Otherwise, \( \sqrt{m} < 2k \) and \( m < 4k^2 \).

Since \( n \leq m \), we have a poly-size kernel.
Consider $k$-MaxLin-AA in the function form:

$$f(x) = \sum_{j=1}^{m} d_j \prod_{i \in I_j} x_i, \ n \leq m.$$  

- Suppose that $r = \max\{|I_j| : j \in [m]\}$ is a constant.
- Then by HI, $\mathbb{E}[f^4] \leq 9^r \mathbb{E}[f^2]^2$.
- By Parseval’s Identity, $\mathbb{E}[f^2] = \sum_{j=1}^{m} d_j^2 \geq m$.
- By the inequality of Alon et al., $\text{Prob}[f \geq \sqrt{m}/(2 \cdot 3^r)] > 0$.
- If $\sqrt{m}/(2 \cdot 3^r) \geq 2k$ then $\text{YES}$. Otherwise, $\sqrt{m}/(2 \cdot 3^r) < 2k$ and $m = O(k^2)$. Since $n \leq m$, we have a poly-size kernel.

We can make use of the Dual HI as well.
We’ve been able to prove the existence of poly-size kernels but only for some special cases of $k$-MaxLin-AA.

The ‘asymmetric’ cases can extended but will still fall far short of the general case.

Open Question: Does $k$-MaxLin-AA admit a poly-size kernel?

We do not know the answer to the question, but we can prove that $k$-MaxLin-AA has a kernel.

Another approach is needed.
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Algorithm $\mathcal{H}$

Introduced by Crowston, Gutin, Jones, Kim and Ruzsa (SWAT’10).
Consider the ‘product form.’

While the system $S$ is nonempty do the following:
1. Choose an arbitrary equation $\prod_{i \in I} x_i = b$ and mark an arbitrary variable $x_\ell$ such that $\ell \in I$.
2. Mark this equation and delete it from the system.
3. Replace every equation $\prod_{i \in I'} x_i = b'$ in the system containing $x_\ell$ by $\prod_{i \in I \Delta I'} x_i = bb'$ (the weight of the equation is unchanged).
4. Apply Reduction Rule 1 to the system.
Lemma (Crowston, Gutin, Jones, Kim and Ruzsa, SWAT 2010)

Let $S$ be an irreducible system and assume that Algorithm $\mathcal{H}$ marks equations of total weight $w$. If $w \geq 2k$ then $S$ is a Yes-instance of $k$-MaxLin-AA.

How to choose equations to mark s.t. $w$ is as large as possible?
Sum-Free Sets

Let $K$ and $M$ be sets of vectors in $\mathbb{F}_2^n$ such that $K \subseteq M$.

- $K$ is $M$-sum-free if no sum of two or more vectors in $K$ is equal to a vector in $M$.

The $M$-sum-free lemma:

**Lemma (Crowston, Gutin, Jones, Kim and Ruzsa, SWAT 2010)**

Let $M$ be a proper subset in $\mathbb{F}_2^n$ such that $\text{span}(M) = \mathbb{F}_2^n$ and $0 \in M$. If $k$ is a positive integer and $k + 1 \leq |M| \leq 2^{n/k}$ then, in time $|M|^{O(1)}$, we can find an $M$-sum-free subset $K$ of $M$ s.t. $|K| = k + 1$. 
Main Technical Theorems

The $M$-sum-free lemma implies Th. 1:

**Theorem (Crowston, Gutin, Jones, Kim, Ruzsa, SWAT 2010)**

Let $S$ be an irreducible system of $k$-MaxLin-AA and let $k \geq 1$. If $2k \leq m \leq 2^n/(2^{k-1}) - 2$, then the answer to $k$-MaxLin-AA is YES. Moreover, we can find a YES-assignment to the variables in time $m^{O(1)}$.

Using Algorithm $\mathcal{H}$ and a depth-bounded search we can prove Th. 2:

**Theorem (Crowston, Fellows, Gutin, Jones, Rosamond, Thomasse, Yeo, arXiv 2011)**

There exists an $n^{2k}(nm)^{O(1)}$-time algorithm for $k$-MaxLin-AA.
**k-MaxLin-AA is FPT**

**Theorem (Crowston, Fellows, Gutin, Jones, Rosamond, Thomasse, Yeo, arXiv 2011)**

\(k\text{-MaxLin-AA has a kernel with at most } O(k^2 \log k) \text{ variables.}\)

**Proof:** Irreducible system with \(m\) equations and \(n\) variables; \(n \leq m\). Cases:

1. \(m < 2k\): \(n = O(k^2 \log k)\).
2. \(2k \leq m \leq 2^{n/(2^k-1)} - 2\): by Th. 1, the answer is YES.
3. \(m \geq n^{2k}\): by Th. 2, we can solve the problem in poly-time.
4. \(2^{n/(2^k-1)} - 1 \leq m \leq n^{2k} - 1\): \(n^{2k} \geq 2^{n/(2^k-1)}\) implying \(n = O(k^2 \log k)\).
Parameterized Algorithm for $k$-MaxLin-AA

**Theorem (Crowston, Fellows, Gutin, Jones, Rosamond, Thomasse, Yeo, arXiv 2011)**

$k$-MaxLin-AA can be solved in time $2^{O(k \lg k)} (nm)^{O(1)}$.

**Proof:** Irreducible system $S$ with $m$ equations and $n$ variables.
1. By the previous theorem, in time $(nm)^{O(1)}$, we either solve $k$-MaxLin-AA for $S$ or get a kernel with $O(k^2 \log k)$ variables.
2. In the last case, apply the $n^{2k} (nm)^{O(1)}$-time algorithm for $k$-MaxLin-AA, which for $n = O(k^2 \log k)$ has runtime $2^{O(k \lg k)} m^{O(1)}$. 
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(k, r)-MaxLin-AA

(k, r)-Max-r-Lin-AA is k-MaxLin-AA in which every equation has at most r variables and k + r is the parameter.

Lemma (Crowston, Fellows, Gutin, Jones, Rosamond, Thomasse, Yeo, arXiv 2011)

Let \( M \subseteq \mathbb{F}_2^n \) s.t. \( \text{span}(M) = \mathbb{F}_2^n \); each vector in \( M \) contains \( \leq r \) non-zero coordinates. If \( n \geq r(k - 1) + 1 \), then in time \( |M|^{O(1)} \), we can find an \( M \)-sum-free subset \( K \) of \( M \) such that \( |K| = k \).

Theorem (ditto)

\((k, r)\)-Max-r-Lin-AA has a kernel with \( \leq (2k - 1)r \) variables.

This improves a kernel with \( n \leq r(r + 1)k \) by Kim and Williams (arXiv 2010).

Open Que.: Is there a poly-size kernel for \((k, r)\)-Max-r-Lin-AA?
Max-$r$-Sat-AA

- CNF formula $F$ with clauses of sizes $r_1, \ldots, r_m$ and variables $y_1, \ldots, y_n$. Let $\max_i r_i \leq r$, a constant.
- $\text{sat}(F, a) =$ the number of clauses satisfied by an assignment $a : \{y_1, \ldots, y_n\} \rightarrow \{\text{true, false}\}$.
- Random assignment $a$. $E := \mathbb{E}[\text{sat}(F, a)] = \sum_{i=1}^{m} (1 - 2^{-r_i})$.
- Max-$r$-Sat-AA: Is there an assignment $a$ s.t. $\text{sat}(F, a) \geq E + k$ ($k$ parameter)?
- Mahajan et al. (2006, 2009): What is the complexity of this problem?
For simplicity: each $r_i = r$.

Let $C$ be a clause of $F$ with variables $y_{p_1}, \ldots, y_{p_r}$.

$$f_C(x) = 1 - \prod_{i=1}^{r}(1 + \varepsilon_{p_i}x_{p_i}), \ x_{p_i} \in \{-1, 1\}, \text{ coef's } \varepsilon_{p_i} \in \{-1, 1\} \text{ and } \varepsilon_{p_i} = 1 \text{ iff } y_{p_i} (\text{not } \bar{y}_{p_i}) \text{ is in } C. \text{ } (y_j = \text{true iff } x_{p_i} = -1.)$$

$$f(x) = \sum_{C \in F} f_C(x).$$

For an assignment $a$, we have $f(x) = 2^r[sat(F, a) - E]$. Thus, $\text{YES iff } f(x) \geq k2^r.$
Max-$r$-Sat-AA Has Quadratic Kernel

- After algebraic simplification: $f(x) = \sum_{J \in \mathcal{F}} c_J \prod_{j \in J} x_i$, a Fourier expansion of $f$, where $|J| \leq r$ for each $J \in \mathcal{F}$.
- Use SABEM [Alon, Gutin, Kim, Szeider, Yeo, SODA 2010] to get either \text{YES} or $m = O(k^2)$.
- This can be extended to CSPs: given $n$ variables and $m$ Boolean formulas, each on at most $r$ variables, define $E$, the average number of the formulas that can be satisfied, determine whether we can satisfy at least $E + k$ of the formulas.
- This problem is FPT. [ditto]
Betweenness

- Let $V = \{v_1, \ldots, v_n\}$ be a set of variables and let $C$ be a set of $m$ betweenness constraints of the form $(v_i, \{v_j, v_k\})$.

- Given a bijection $\alpha : V \rightarrow \{1, \ldots, n\}$, we say that a constraint $(v_i, \{v_j, v_k\})$ is satisfied if either $\alpha(v_j) < \alpha(v_i) < \alpha(v_k)$ or $\alpha(v_k) < \alpha(v_i) < \alpha(v_j)$.

- **Betweenness**: find a bijection $\alpha$ satisfying the max number of constraints in $C$.

- Tight Lower Bound: $m/3$, the expected number of satisfied constraints.

- Charikar, Guruswami and Manokaran (CCC’09): Approximating **Betweenness** within factor $1/3 + \varepsilon$ is Unique-Games hard.
Betweenness-AA

- Betweenness-AA: Is there $\alpha$ that satisfies $\geq m/3 + \kappa$ constraints? ($\kappa$ is the parameter)
- Benny Chor’s question in Niedermeier’s book (2006): What is the parameterized complexity of Betweenness-AA?
- Reduction Rule: delete complete triples $(1, \{2, 3\}), (2, \{3, 1\}), (3, \{1, 2\})$.
- We can introduce $X$ required by SABEM, but ...
- It’s difficult to estimate $\mathbb{E}(X^2)$, practically impossible to do $\mathbb{E}(X^4)$, but we cannot use Hypercontractive Inequality as $X$ is not a polynomial of constant-bounded degree.
Introducing Four Bins

- An instance $(V, C)$, where $V$ is the set of variables and $C = \{C_1, \ldots, C_m\}$ is the set of betweenness constraints.
- A function $\phi : V \rightarrow \{0, 1, 2, 3\}$ (vertices into 4 bins).
- $\phi$-compatible bijections $\alpha$: if $\phi(v_i) < \phi(v_j)$ then $\alpha(v_i) < \alpha(v_j)$. 

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More Applications of Linear Algebra, Probabilistic Method and Hypercontractive Inequality

Using Four Bins

- Let $\alpha$ be a random $\phi$-compatible bijection and $\nu_p(\alpha) = 1$ if $C_p$ is satisfied and 0, otherwise.
- Let the weights $w(C_p, \phi) = E(\nu_p(\alpha)) - 1/3$ and $w(C, \phi) = \sum_{p=1}^{m} w(C_p, \phi)$.

Lemma

If $w(C, \phi) \geq \kappa$ then $(V, C)$ is a YES-instance of Betweenness-AA.

- Thus, to solve Betweenness-AA, it suffices to find $\phi$ for which $w(C, \phi) \geq \kappa$.
- We may forget about bijections $\alpha$ and use SABEM!
Exploiting Four Bins

- We can get $X$ of degree 6.
- $\mathbb{E}[X^2] \geq 11m/768$.
- The rest is easy.
- Gutin, Kim, Mnich and Yeo, JCSS 2011: Betweenness-AA has an $O(\kappa^2)$-kernel.
Thank you!

- Questions?
- Comments?