A Lie algebraic classification of continuous-time Markov models

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Joint work with Jesús Fernández-Sánchez and Peter Jarvis

Phylogenetics: New data, new Phylogenetic challenges
Follow-up Meeting, June 2011

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- Just a matter of finding (stochastic) subalgebras of GMM?

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- Orbit/stablizer thm: Any orbit is isomorphic to $\mathcal{G}/\mathcal{G}_x$, $\mathcal{G}_x < \mathcal{G}$.
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$F81$	4	$\mathfrak{S}_4/\mathfrak{S}_3$	$id \oplus \{31\}$	$[R_i - R_j] = R_i - R_j$
$K3ST$	3	$\mathfrak{S}_4/\left(\mathbb{Z}_2 wr \mathbb{Z}_2\right)$	$id \oplus \{2^2\}$	$[L_x, L_y] = 0$
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- We are still organizing the list of Lie Markov models for the case of $\mathbb{Z}_2 wr \mathbb{Z}_2$ symmetry...

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Thanks and please consider coming to Phylomania 2011.