Arithmetic and physics of Higgs moduli spaces

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Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; fix group GL_n

$$\mathcal{M}^d_{\mathrm{Dol}} := \left\{ \begin{array}{c} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles } (E,\phi) \\ \phi \in H^0(C, End(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{B}} := \{A_1, B_1, .., A_g, B_g \in \mathrm{GL}_n | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id\} /\!\!/ \mathrm{PGL}_n$$

when $(d,n) = 1$ these are smooth non-compact varieties

- Non-Abelian Hodge Theorem: $\mathcal{M}^d_{\mathrm{Dol}} \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{B}}$ (Hitchin, Donaldson, Corlette, Simpson)
- $g = 1 \rightsquigarrow$ Stone-von Neumann $\rightsquigarrow$ $\mathcal{M}^d_{\mathrm{B}} \cong (\mathbb{C}^*)^2 \cong T^*\mathrm{Jac}(C) \cong \mathcal{M}^d_{\mathrm{Dol}}$
- Problem: what is Poincaré polynomial $P(\mathcal{M}^d_{\mathrm{Dol}}; t) = P(\mathcal{M}^d_{\mathrm{B}}; t)$?

Mixed Hodge polynomials

- (Deligne 1971) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X;\mathbb{Q})$ for any complex algebraic variety X
- $H(X;q,t) = \sum \dim(W_i/W_{i-1}(H^k(X)))t^k q^{\frac{i}{2}}$, *mixed Hodge polynomial*
- $P(X;t) = H(X;1,t)$, *Poincaré polynomial*
- $E(X;q) = q^d H(1/q,-1)$, *E-polynomial* of X.

Theorem (Katz 2008)

If M is a smooth quasi-projective variety defined over $\mathbb{Z}$ and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q, then $E(M;q) = E(q)$.

Mixed Hodge polynomials

- (Hausel-Villegas 2008) calculates
  $E(\mathcal{M}_{\mathrm{B}}^{d};q)=|\mathcal{M}_{\mathrm{B}}^{d}(\mathbb{F}_q)|=\sum_{\chi\in Irr(\mathrm{GL}_n(\mathbb{F}_q))}\frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}}\frac{\chi(\xi_n^d)}{\chi(1)}$
- $\rightsquigarrow E(\mathcal{M}_{\mathrm{B}}^{d};q)=E(\mathcal{M}_{\mathrm{B}}^{d'};q)$ when $(d,n)=(d',n)=1$
- $\mathcal{M}_{\mathrm{B}}^{d}$ and $\mathcal{M}_{\mathrm{B}}^{d'}$ Galois conjugate $\Rightarrow H(\mathcal{M}_{\mathrm{B}}^{d};q,t)=H(\mathcal{M}_{\mathrm{B}}^{d'};q,t)$

Conjecture (Hausel-Villegas, 2008)

$$\sum_\lambda \prod \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2g}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})}\, T^{|\lambda|}=\exp\left(\sum_{n,k}\frac{H\left(\mathcal{M}_{\mathrm{B}}^{d};w^{2k},-(zw)^{-2k}\right)(zw)^{dn}}{(z^{2k}-1)(1-w^{2k})}\frac{T^{nk}}{k}\right)$$

- when $g=1$ $\mathcal{M}_{\mathrm{B}}^{d}=(\mathbb{C}^*)^2$ by Stone-von Neumann $\overset{HV}{\rightsquigarrow}$

$$\sum_q \prod \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})}\, T^{|q|}=\exp\left(\sum_{k\geq 1}\frac{(z^k-w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)}\frac{T^k}{k}\right)$$

Curious Hard Lefschetz

- (Hausel-Villegas 2008) calculates
  $E(\mathcal{M}_{\mathrm{B}}^d; q) = |\mathcal{M}_{\mathrm{B}}^d(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$
- we find $E(\mathcal{M}_{\mathrm{B}}^d; q) = q^{d_n} E(\mathcal{M}_{\mathrm{B}}^d; 1/q)$ palindromic
  by *Alvis-Curtis duality*

$$q^{\frac{n(n-1)}{2}} \chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi, \chi' \in Irr(\mathrm{GL}_n(\mathbb{F}_q))$$

- $\rightsquigarrow$ Curious Hard Lefschetz Conjecture (theorem when $n = 2$):

$$\begin{array}{cccc} L^l: & Gr^W_{d_n-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}}^d)) & \to & Gr^W_{d_n+2l} H^{i+l}(\mathcal{M}_{\mathrm{B}}^d) \\ & x & \mapsto & x \cup \alpha^l \end{array},$$

  where $\alpha \in W_4 H^2(\mathcal{M}_{\mathrm{B}}^d)$
- The implied functional equation on the conjectured
  $H(\mathcal{M}_{\mathrm{B}}^d; q, t) = (qt)^{d_n} H(\mathcal{M}_{\mathrm{B}}^d; \frac{1}{qt^2}, t)$ holds

Perverse filtration

- $f : X \to Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration* $P_0 \subset \cdots \subset P_i \subset \ldots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $Rf_*(\mathbb{Q}_X)$ into perverse sheaves
- recipe (de Cataldo-Migliorini, 2008) for perverse filtration when X smooth and Y affine:
  take $Y_0 \subset \cdots \subset Y_i \subset \ldots Y_d = Y$
  s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X)} \to \mathrm{H^k(f^{-1}(Y_i))})$$

- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{cccc} L^l : & Gr^P_{d-l}(H^*(X)) & \to & Gr^P_{d+l}H^{*+2l}(X) \\ & x & \mapsto & x \cup \alpha^l \end{array}$$

  where $\alpha \in H^2(X)$ is a relative ample class

Main conjecture

- recall Hitchin map $\begin{array}{cccc} \chi : & \mathcal{M}^d_{\mathrm{Dol}} & \to & \mathbb{A} := \bigoplus_{i=1}^n H^0(C;K^i) \\ & (E,\phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$
- (Hitchin 1987) → completely integrable Hamiltonian system and *proper*

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}^d_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}^d_{\mathrm{B}})$ *under the isomorphism* $H^*(\mathcal{M}^d_{\mathrm{Dol}}) \cong H^*(\mathcal{M}^d_{\mathrm{B}})$ *from non-Abelian Hodge theory.*
In particular CHL$\Leftrightarrow$ *RHL*

Theorem (de Cataldo-Hausel-Migliorini 2010)

$P = W$ *for* $n = 2$.

- proof mirroring Ngô's proof of the fundamental lemma
- evidence for $n > 2$?

Refined Gopakumar-Vafa conjecture for local curves

- we follow (Chuang-Diaconescu-Pan 2011)
- Y total space of $\mathcal{O}_C \oplus K_C$ over C
- Y CY 3-fold , "local curve"
- conjectural "quantum" Pandharipande-Thomas invariants

  $Z_{PT}^{ref} := \sum_{\beta \in H_2(Y)} \sum_{n \in \mathbb{Z}} T^\beta q^e E_{virt}(\mathcal{P}(Y, \beta, e); y)$
- Gopakumar-Vafa generating function of refined BPS invariants:

$$F_{GV}^{ref} := \sum_{\substack{k \geq 1 \\ \beta \in H_2(Y)}} \sum_{j_L, l \geq 0} \frac{T^{k\beta}}{k} (-1)^{j_L + l} N_\beta^{(j_L, l)} \frac{(q^{-kj_L} + \cdots + q^{kj_L}) q^{-k} y^l}{(1-(qy)^{-k})(1-(q/y)^{-k})}$$

Conjecture ("refined BPS", Chuang-Diaconescu-Pan 2011)

$$Z_{PT}^{ref} = \exp\left(F_{GV}^{ref}\right)$$

F_{GV}^{ref} via Relative Hard Lefschetz

- Gopakumar-Vafa's BPS invariants $N_\beta^{j_L,j_R}$ heuristically arise from decomposing the cohomology $H^*(\mathcal{M}_\beta^e)$ of the space of D-branes via a putative action of $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$
- for local curve Y, (Chuang-Diaconescu-Pan 2011) argue that $\mathcal{M}_\beta^e \cong \mathcal{M}_{\mathrm{Dol}}^d$ where $\beta = n[C]$ and $d = e + n(g-1)$
- recall $\chi : \mathcal{M}_{\mathrm{Dol}}^d \to \mathbb{A}$ induces perverse filtration P on $H^*(\mathcal{M}_{\mathrm{Dol}}^d)$ with RHL
- RHL on $\mathrm{Gr}^P(H^*(\mathcal{M}_{\mathrm{Dol}}^d)) \rightsquigarrow (\mathfrak{sl}_2)_L$ action on $\mathrm{Gr}^P(H^*(\mathcal{M}_{\mathrm{Dol}}^d))$
- the corresponding primitive decomposition $H^m(\mathcal{M}_{\mathrm{Dol}}^d) \cong \oplus_{i,j} Q^{i,j;m}$ gives at least a $(\mathfrak{gl}_1)_R$ action
- Chuang-Diaconescu-Pan define $N_\beta^{j_L;l} := \dim(Q^{j_L,0;l})$

Z^{ref}_{PT} via geometric engineering

- geometric engineering $\rightsquigarrow Z^{ref}_{PT} = Z_{gauge}$
  Z_{gauge} partition function of certain gauge theory
- Chuang-Diaconescu-Pan argue that it should be a $U(1)$-gauge theory on $X = \mathbb{R}^4 \cong \mathbb{C}^2$
- $Z_{gauge} = \sum_{k\geq 0} Q^k \chi^{\mathbb{T}^2}_{y'} \left(\det(\mathcal{V}_k)^{1-g} \otimes (T^* X^{[k]})^{\oplus g}\right)$

  $\mathcal{V}_k$ is the tautological bundle on the Hilbert scheme $X^{[k]}$
  $\mathbb{T}^2$ acts on X and so on $X^{[k]}$ with isolated fixed points
  $\rightsquigarrow Z_{gauge}$ is defined by localizing to the fixed points
- after changes of variables we have

$$\begin{array}{ccccc} Z_{gauge} & \overset{g.e.}{=} & Z^{ref}_{PT} & \overset{CDP}{=} & \exp(F_{GV}) \\ \| & & & & \|_{P=W} \\ \sum_\lambda \prod \frac{\left(z^{2l+1}-w^{2a+1}\right)^{2g}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} & & \overset{HV}{=} & & \exp\left(\sum_{n,k} \frac{H\left(\mathcal{M}^d_{\mathrm{B}}; w^{2k}, -(zw)^{-2k}\right)(zw)^{d_n}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{nk}}{k}\right) \end{array}$$

- conclusion: HV, CDP + $g.e.$ $\Rightarrow P = W$

$g = 1$ via geometric engineering

- when $g = 1$ $\mathcal{M}_{\mathrm{B}}^{d} = (\mathbb{C}^{\times})^{2}$ by Stone-von Neumann $\overset{HV}{\rightsquigarrow}$

$$\sum_{\lambda} \prod \frac{\left(z^{2l+1} - w^{2a+1}\right)^{2}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp\left(\sum_{k \geq 1} \frac{(z^{k} - w^{k})^{2}}{(z^{2k} - 1)(1 - w^{2k})(1 - T^{k})} \frac{T^{k}}{k} \right)$$

- after geometric engineering this formula becomes

$$\sum_{n} \chi_{y}^{\mathbb{T}^{2}}(X^{[n]}) T^{n} = \sum_{n} \chi_{y,st}^{\mathbb{T}^{2}}(X^{n}/S_{n}) T^{n}$$

(Waelder, 2008) proves geometrically a more general DMVV formula for equivariant elliptic genus $\rightsquigarrow$ our $g = 1$ formula follows!

Mathematical outlook

- (Chuang-Diaconescu-Pan 2011) refined BPS conjecture $\Leftrightarrow$ (Hausel-Villegas 2008) conjecture on $H(\mathcal{M}^d_{\mathrm{B}}; q, t)$ provided $P = W$ of (de Cataldo-Hausel-Migliorini 2010)
- studying wall-crossing for the stability condition for $Z^{ref}_{PT}(Y) \rightsquigarrow$ recursive formulas (Chuang-Diaconescu-Pan 2010) for $P_t(\mathcal{M}^d_{\mathrm{Dol}})$ studied by (Mozgovoy 2011)
- mathematically geometric engineering proposes a deep connection between $K_{\mathbb{T}^2}((\mathbb{C}^2)^{[n]})$ and $H^*(\mathcal{M}^d_{\mathrm{Dol}})$
- may lead to connections between (Haiman 2002) and (Hausel-Letellier-Villegas 2008)
  explaining the appearance of Macdonald polynomials in both
- DAHA acts on $K_{\mathbb{T}^2}((\mathbb{C}^2)^{[n]})$ by (Gordon-Stafford 2004)
  it is expected that DAHA acts on $H^*(\mathcal{M}^d_{\mathrm{Dol}})$ from (Yun 2009)
- are these DAHA actions related by geometric engineering?