Transmission Eigenvalues in Inverse Scattering Theory

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What are Transmission Eigenvalues?

Scattering by an Inhomogeneous Media

\[ \Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^2 \]
\[ u = u^s + u^i \quad \text{in } \mathbb{R}^2 \]
\[ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \]

We assume that \( n - 1 \) has compact support \( \overline{D} \) and \( n \in L^\infty(D) \) is such that \( \Re(n) \geq \gamma > 0 \) and \( \Im(n) \geq 0 \) in \( \overline{D} \).

**Question**: Is there an incident wave \( u^i \) that does not scatter?

The answer to this question leads to the transmission eigenvalue problem.
Transmission Eigenvalue Problem

If there exists a nontrivial solution to the homogeneous interior transmission problem

\[
\Delta w + k^2 n(x) w = 0 \quad \text{in} \quad D
\]
\[
\Delta v + k^2 v = 0 \quad \text{in} \quad D
\]
\[
w = v \quad \text{on} \quad \partial D
\]
\[
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D
\]

such that \( v \) can be extended outside \( D \) as a solution to the Helmholtz equation \( \tilde{v} \), then the scattered field due to \( \tilde{v} \) as incident wave is identically zero.

**Remark:** Note that if \( n = 1 \) the interior transmission problem is degenerate.
**Transmission Eigenvalue Problem**

**Definition:** $k^2 \in \mathbb{C}$ is a transmission eigenvalue if there exists a nontrivial solution $v \in L^2(D), w \in L^2(D), w - v \in H^2_0(D)$ of the homogeneous interior transmission problem

\[
\begin{align*}
\Delta w + k^2 n(x) w &= 0 & \text{in} & & D \\
\Delta v + k^2 v &= 0 & \text{in} & & D \\
w &= v & \text{on} & & \partial D \\
\frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} & \text{on} & & \partial D
\end{align*}
\]

Corresponding nontrivial solutions $(v, w)$ are called eigenpairs

**Note:** If $\Im(n) > 0$ in $\overline{D}$, there are no real transmission eigenvalues.
Motivation

Two important issues:

- Real transmission eigenvalues can be determined from the scattered data.
- Transmission eigenvalues carry information about material properties.

Therefore, transmission eigenvalues can be used to quantify the presence of abnormalities inside homogeneous media and use this information to test the integrity of materials.

Let us start by assuming that $\Im(n) = 0$ in $\overline{D}$.

How are real transmission eigenvalues seen in the scattering data?
Measurements

We assume that $u^i(x) = e^{ikx \cdot d}$ and the far field pattern $u_\infty(\hat{x}, d, k)$ of the scattered field $u^s(x, d, k)$ is available for $\hat{x}, d \in \Omega$, and $k \in [k_0, k_1]$

where $u^s(x, d, k) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d, k) + O\left(\frac{1}{r^{3/2}}\right)$

as $r \to \infty$ where $\hat{x} = x/|x|$, $r = |x|$, $k > 0$ is the wave number and $\Omega$ is the unit circle.

Define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d, k)g(d)ds(d), \quad \left( S = I + \frac{ik}{\sqrt{2\pi k}}e^{-i\pi/4}F \right)$$
The Far Field Equation

From now on we assume that $D$ (or a reconstruction of $D$) is known. For $z \in D$ the far field equation is

$$(Fg)(\hat{x}) = \Phi_{\infty}(\hat{x}, z, k), \quad g \in L^2(\Omega)$$

where $\Phi_{\infty}(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{4\pi k}}e^{-ik\hat{x} \cdot z}$ is the far field pattern of the fundamental solution $\Phi(x, z, k) := 1/4H_0^{(1)}(k|x - z|)$.

In fact only the measured "noisy" far field pattern $u_{\infty}^\delta(\hat{x}, d, k)$ is available, where $\delta > 0$ is the noise level, which leads to the noisy far field equation

$$(F^\delta g)(\hat{x}) := \int_{\Omega} u_{\infty}^\delta(\hat{x}, d, k)g(d)ds(d) = \Phi_{\infty}(\hat{x}, z, k).$$
Computation of Real TE

Theorem: Assume that either $n > 1$ or $n < 1$ for $x \in \overline{D}$. For $z \in D$, let $g_{z,\delta,k}$ be the Tikhonov regularized solution of the far field equation

$$(F^\delta g)(\hat{x}) = \Phi_\infty(\hat{x}, z, k).$$

- If $k^2$ is not a transmission eigenvalue then

$$\lim_{\delta \to 0} ||v_{g_{z,\delta,k}}||_{L^2(D)}$$

exists.

*Arens (Inverse Problems 2004),*

- If $k^2$ is a transmission eigenvalue then for almost every $z \in D$

$$\lim_{\delta \to 0} ||v_{g_{z,\delta,k}}||_{L^2(D)} = \infty.$$

*Cakoni-Colton-Haddar (Comptes Rendus Math. 2010).*
Computation of Real TE

A composite plot of $\|g_{z_i}\|_{L^2(\Omega)}$ against $k$
for 25 random points $z_i \in D$

The average of $\|g_{z_i}\|_{L^2(\Omega)}$
over all choices of $z_i \in D$.

Computation of the transmission eigenvalues from the far field equation
for the unit square $D$. 
Historical Overview

- The transmission eigenvalue problem in scattering theory was introduced by Kirsch (1986) and Colton-Monk (1988).


In the above work, it is always assumed that either $n - 1 > 0$ or $1 - n > 0$. 
The first proof of existence of at least one transmission eigenvalue for large enough contrast is due to Päivärinta-Sylvester (2009).

The existence of an infinite set of transmission eigenvalues is proven by Cakoni-Gintides-Haddar (2010) under only assumption that either \( n - 1 > 0 \) or \( 1 - n > 0 \). The existence has been extended to other scattering problems by Cakoni-Kirsch (2010), Cakoni-Cossonniere-Haddar (to appear) etc.

Hitrik-Krupchyk-Ola-Päivärinta (2010), in a series of papers have extended the transmission eigenvalue problem to a more general class of differential operators with constant coefficients.
Finch has connected the discreteness of the transmission spectrum to a uniqueness question in thermo-acoustic imaging for which \( n - 1 \) could change sign.

Cakoni-Colton-Haddar (2010) have investigated the case when \( n = 1 \) in \( D_0 \subset D \) and \( n - 1 > \alpha > 0 \) in \( D \setminus \overline{D_0} \).

Recently Sylvester (to appear) has shown that the set of transmission eigenvalues is at most discrete requiring that \( n - 1 \) is positive (or negative) only in a neighborhood of \( \partial D \) but otherwise could change sign inside \( D \). A similar result is obtained by Bonnet Ben Dhia - Chesnel - Haddar (to appear) using T-coercivity for the case when there is contrast in both the main differential operator and lower term.
Transmission Eigenvalues

We assume that $n \in L^\infty(D)$ is such that $n - 1 \geq \alpha > 0$.

The transmission eigenvalue problem can be written for the difference $u := w - v \in H^2_0(D)$ as an eigenvalue problem for the fourth order equation:

$$(\Delta + k^2) \frac{1}{n - 1}(\Delta + k^2 n) u = 0$$

i.e. in the variational form

$$\int_D \frac{1}{n - 1} (\Delta u + k^2 nu)(\Delta \varphi + k^2 \varphi) \, dx = 0 \quad \text{for all} \ \varphi \in H^2_0(D)$$
Transmission Eigenvalues

Letting \( k^2 := \tau \), the transmission eigenvalue problem can be written as a quadratic pencil operator

\[ u - \tau K_1 u + \tau^2 K_2 u = 0, \quad u \in H^2_0(D) \]

with selfadjoint compact operators \( K_1 = T^{-1/2} T_1 T^{-1/2} \) and \( K_2 = T^{-1/2} T_2 T^{-1/2} \) where

\[
(Tu, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1} \Delta u \Delta \overline{\varphi} \, dx \quad \text{coercive}
\]

\[
(T_1 u, \varphi)_{H^2(D)} = -\int_D \frac{1}{n-1} \left( \Delta u \overline{\varphi} + nu \Delta \overline{\varphi} \right) \, dx
\]

\[
(T_2 u, \varphi)_{H^2(D)} = \int_D \frac{n}{n-1} u \overline{\varphi} \, dx \quad \text{non-negative.}
\]
Transmission Eigenvalues

The transmission eigenvalue problem can be transformed to the eigenvalue problem

$$(\mathbb{K} - \xi \mathbb{I})U = 0, \quad U = \begin{pmatrix} u \\ \tau K_2^{1/2} u \end{pmatrix}, \quad \xi := \frac{1}{\tau}$$

for the non-self-adjoint compact operator

$$\mathbb{K} : H_0^2(D) \times H_0^2(D) \rightarrow H_0^2(D) \times H_0^2(D)$$
given by

$$\mathbb{K} := \begin{pmatrix} K_1 & -K_2^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}.$$

However from here one can see that the transmission eigenvalues form a discrete set with $+\infty$ as the only possible accumulation point.
Transmission Eigenvalues

To obtain existence of transmission eigenvalues and isoperimetric Faber-Krahn type inequalities we rewrite the transmission eigenvalue problem in the form

\[(A_\tau - \tau B)u = 0 \text{ in } H^2_0(D)\]

\[ (A_\tau u, \varphi)_{H^2(D)} = \int_D \frac{1}{n-1} (\Delta u + \tau u)(\Delta \varphi + \tau \varphi) \, dx + \tau^2 \int_D u \cdot \bar{\varphi} \, dx \]

\[ (Bu, \varphi)_{H^2(D)} = \int_D \nabla u \cdot \nabla \bar{\varphi} \, dx \]

Observe that

- The mapping \( \tau \rightarrow A_\tau \) is continuous from \((0, +\infty)\) to the set of self-adjoint coercive operators from \(H^2_0(D) \rightarrow H^2_0(D)\).

- \( B : H^2_0(D) \rightarrow H^2_0(D) \) is self-adjoint, compact and non-negative.
Now we consider the generalized eigenvalue problem

\[(A_\tau - \lambda(\tau)B)u = 0 \quad \text{in} \quad H^2_0(D)\]

Note that \(k^2 = \tau\) is a transmission eigenvalue if and only if \(\lambda(\tau) = \tau\)

For a fixed \(\tau > 0\) there exists an increasing sequence of eigenvalues \(\lambda_j(\tau)_{j \geq 1}\) such that \(\lambda_j(\tau) \to +\infty\) as \(j \to \infty\).

These eigenvalues satisfy

\[
\lambda_j(\tau) = \min_{W \subset \mathcal{U}_j} \left( \max_{u \in W \setminus \{0\}} \frac{(A_\tau u, u)}{(B u, u)} \right).
\]
Transmission Eigenvalues

Hence, if there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that

- $A_{\tau_0} - \tau_0 B$ is positive on $H_0^2(D)$,
- $A_{\tau_1} - \tau_1 B$ is non positive on a $m$ dimensional subspace of $H_0^2(D)$

then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \ldots, m$, has at least one solution in $[\tau_0, \tau_1]$ meaning that there exists $m$ transmission eigenvalues (counting multiplicity) within the interval $[\tau_0, \tau_1]$.

It is now obvious that determining such constants $\tau_0$ and $\tau_1$ provides the existence of transmission eigenvalues as well as isoperimetric inequalities for the first transmission eigenvalue.

In what follows we denote

$$n^* = \sup_{x \in D} n(x) \quad \text{and} \quad n_* = \inf_{x \in D} n(x).$$
Faber-Krahn Inequalities

Theorem: Assume that $n \in L^\infty(D)$, and either $1 < n_* \leq n(x) \leq n^*$ almost everywhere in $\overline{D}$. Then, there exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$. Furthermore

$$k_{1,n^*}^2 \leq k_{1,n(x)}^2 \leq k_{1,n_*}^2.$$

One can prove that, for $n$ constant, the first transmission eigenvalue $k_{1,n}^2$ is continuous and strictly monotonically increasing with respect to $n$. In particular, this shows that the first transmission eigenvalue determine uniquely the constant index of refraction, provided that it is known a priori that $n > 1$.

Similar results can be obtained for the case when $0 < n_* \leq n(x) \leq n^* < 1$.

Detection of Anomalies in an Isotropic Medium

We find the constant $n_0$ such that the first transmission eigenvalue of

\[
\Delta w + k^2 n_0 w = 0 \quad \text{in} \quad D \\
\Delta v + k^2 v = 0 \quad \text{in} \quad D \\
w = v \quad \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D
\]

is $k_{1,n(x)}^2$ (which can be determined from the measure data).

Then from the previous discussion we have that $n_* \leq n_0 \leq n^*$.

Numerical examples show in fact that $n_0 \approx \frac{1}{|D|} \int_D n(x) \, dx$.

Can the assumption $n > 1$ or $0 < n < 1$ in $D$ be relaxed?

Recent results by John Sylvester on the discreteness of transmission eigenvalues assuming the above only in a neighborhood of $\partial D$. 
The Case with Cavities

The case when there are regions in $D$ where $n = 1$ (i.e. cavities) is more delicate.

The same type of analysis can be carried through by looking for solutions of the transmission eigenvalue problem $v \in L^2(D)$ and $w \in L^2(D)$ such that $w - v$ is in

$$V_0(D, D_0, k) := \{ u \in H^2_0(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0 \}.$$

In particular one has the Faber Krahn inequalities:

$$0 < \frac{\lambda_1(D)}{n^*} \leq k^2_{1,D,D_0,n(x)} \leq k^2_{1,B,n^*}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in $D$ and $B$ is the largest ball such that $B \subset D \setminus \overline{D_0}$.
Complex Eigenvalues

Current results on complex transmission eigenvalues for media of general shape are limited to identifying eigenvalue free zones in the complex plane.

- The first result for homogeneous media is given in Cakoni-Colton-Gintides SIMA (2010).
- The best result to date is due Hitrik-Krupchyk-Ola-Päivärinta, (2010) arXiv. In particular they have shown that almost all transmission eigenvalues are confined to a parabolic neighborhood of the positive real axis. More specifically:

**Theorem:** For \( n \in C^\infty(D, \mathbb{R}) \) and \( 1 < \alpha \leq n \leq \beta \), there exists a \( 0 < \delta < 1 \) and \( C' > 1 \) both independent of \( n \) such that all transmission eigenvalues \( \tau := k^2 \in \mathbb{C} \) with \( |\tau| > C \) satisfies \( \Re(\tau) > 0 \) and \( \Im(\tau) \leq C'|\tau|^{1-\delta} \).
Absorbing Medium

We are interested in understanding the interior transmission spectrum in the case of absorbing/dispersive media. In this case

\[
\Delta w + k^2 \left( \epsilon_1 + i \frac{\gamma_1}{k} \right) w = 0 \quad \text{in} \quad D \\
\Delta v + k^2 \left( \epsilon_0 + i \frac{\gamma_0}{k} \right) v = 0 \quad \text{in} \quad D
\]

\[w - v \in H^2_0(D), \text{ where } \epsilon_0 \geq \alpha_0 > 0, \epsilon_1 \geq \alpha_1 > 0, \gamma_0 > 0, \gamma_1 > 0.\]

We have shown that the set of transmission eigenvalues \(k \in \mathbb{C}\) is discrete either

- in the region \(Re(k) \geq \sigma > 0\) provided \(\epsilon_1 - \epsilon_0 \geq \theta > 0\) or

- in the region \(Im(k) \geq -\frac{\sup_D(\gamma_1 - \gamma_0)}{\sup_D(\epsilon_1 - \epsilon_0)}\) provided

\[\epsilon_1 - \epsilon_0 \geq \theta > 0 \text{ and } \gamma_1 - \gamma_0 > 0.\]
Absorbing Media

Furthermore, for small absorption $\gamma_0$ and $\gamma_1$, based on a perturbation argument it is possible to show that there is at least a complex transmission eigenvalue in a neighborhood of the first real transmission eigenvalue corresponding to $\gamma_0 = \gamma_1 = 0$.

For the case of $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\gamma_0 > 0$, $\gamma_1 > 0$ all constant, we have identified eigenvalue free zones in the complex plane $k \in \mathbb{C}$ Cakoni-Colton-Haddar (to appear).

In particular, all real transmission eigenvalues satisfy

$$k^2 \geq \lambda_1(D) \frac{\gamma_0 + \gamma_1}{\gamma_0 \epsilon_1 + \gamma_1 \epsilon_0},$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue for $-\Delta$ in $D$.

The existence of transmission eigenvalues for general media if absorption is present is still open.
Anisotropic Media

The corresponding transmission eigenvalue problem is to find \( v, w \in H^1(D) \) such that

\[
\begin{align*}
\nabla \cdot A \nabla w + k^2 nw &= 0 \quad \text{in} \quad D \\
\Delta v + k^2 v &= 0 \quad \text{in} \quad D \\
w &= v \quad \text{on} \quad \partial D \\
\nu \cdot A \nabla w &= \nu \cdot \nabla v \quad \text{on} \quad \partial D
\end{align*}
\]

Cakoni-Kirsch, IJCSM (2010).

We define

\[
a^* = \sup_{x \in D} \sup_{|\xi| = 1} (\xi \cdot A(x) \xi), \quad a_* = \inf_{x \in D} \inf_{|\xi| = 1} (\xi \cdot A(x) \xi)
\]

\[
n^* = \sup_{x \in D} n(x) \quad \text{and} \quad n_* = \inf_{x \in D} n(x).
\]
Anisotropic Media

Assume that either $a_*$ and $0 < n^* < 1$ or $n(x) \equiv 1$, or $0 < a^* < 1$ and $n_* > 1$ or $n(x) \equiv 1$.

There exists an infinite discrete set of real transmission eigenvalues accumulating at $+\infty$.

The following Faber-Krahn inequalities hold:

- If $a_* > 1$ and $0 < n^* < 1$ then

$$0 < \lambda_1(D) \leq k_{1,D,A,n}^2 \leq k_{1,B,a*,n}^2.$$ 

- If $0 < a^* < 1$ and $n_* > 1$ then

$$0 < \frac{a_*}{n_*} \lambda_1(D) \leq k_{1,D,A,n}^2 \leq k_{1,B,a*,n_*}^2.$$ 

where the $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in $D$ and $B$ is the largest ball such that $B \subset D$ ($B$ could possibly be $D$).
Anisotropic Media

In the opposite case, i.e. either $a_\ast > 1$ and $n_\ast > 1$, or $0 < a_\ast < 1$ and $0 < n_\ast < 1$:

- The set of transmission eigenvalues is discrete with $+\infty$ as the only accumulation point.

  Cakoni-Colton-Haddar (2009)

- There exists at least one real transmission eigenvalue providing that $n_\ast$ is small enough.

  Cakoni-Kirsch, IJCSM (2010)

The analysis has been extended to the case when $D$ contains perfectly conducting subregions (Dirichlet inclusions) and/or cracks by Anne Cossonniere in her Ph.D. thesis, Cakoni-Cossoniere-Haddar (to appear).
In the case of \( n(x) = 1 \) we have the isoperimetric inequality

\[
k_{1,a\ast}^2 \leq k_{1,A(x)}^2 \leq k_{1,a\ast}^2, \quad \text{if } 0 < a^\ast < 1.
\]

Note nonuniqueness of the determination of \( A(x) \) from scattering data.

Given the first transmission eigenvalue \( k_{1,A(x)}^2 \) and the domain \( D \) our aim is to obtain information about \( A(x) \). We find \( a_0 \) such that the first transmission eigenvalue corresponding to

\[
a_0 \Delta w + k^2 w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in} \quad D
\]

\[
w = v, \quad a_0 \, \nu \cdot \nabla w = \nu \cdot \nabla v \quad \text{on} \quad \partial D.
\]

equals the measured transmission eigenvalue. The above monotonicity now gives \( a_\ast \leq a_0 \leq a^\ast \) i.e. \( a_0 \) lies between the infimum of the smallest eigenvalue and the supremum of the largest eigenvalue of \( A(x) \).
Numerical Examples: Homogeneous Anisotropic Media

We consider $D$ to be the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$ and

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2 = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \quad A_{2r} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}$$

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Eigenvalues $a_<em>, a^</em>$</th>
<th>Predicted $a_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{iso}$</td>
<td>4, 4</td>
<td>4.032</td>
</tr>
<tr>
<td>$A_1$</td>
<td>2, 8</td>
<td>5.319</td>
</tr>
<tr>
<td>$A_2$</td>
<td>6, 8</td>
<td>7.407</td>
</tr>
<tr>
<td>$A_{2r}$</td>
<td>6, 8</td>
<td>6.896</td>
</tr>
</tbody>
</table>

Bonnet Ben Dhia - Chesnel - Haddar (to appear) have used the concept of $\top$-coercivity to investigate the transmission eigenvalue problem for the case of contrasts changing sign. In particular the have obtained the following results:

Assume that either $a_\ast > 1$ and $n_\ast > 1$, or $0 < a_\ast < 1$ and $0 < n_\ast < 1$ in a neighborhood $\mathcal{N}$ of $\partial D$. Then the set of transmission eigenvalues is at most discrete. Furthermore, there exist two positive constants $\rho > 0$ and $\delta > 0$ such that $k \in \mathbb{C}$ satisfying $|k| > \rho$ and $|\Re(k)| < \delta|\Im(k)|$ are not a transmission eigenvalues.

Note that there are no assumptions on the sign of $A - I$ and $n - 1$ in $D \setminus \mathcal{N}$. 
Contrasts Changing Sign

In addition, in the same paper it is shown

- Assume that \( \int_D (n - 1) \, dx \neq 0 \) and either \( 0 < a^* < 1 \) or \( a^* > 1 \).

Then the set of transmission eigenvalues is at most discrete.

- Furthermore, the Faber-Krahn inequality for the transmission eigenvalue \( k_1 \) of the smallest amplitude holds

\[
|k_1|^2 \geq \frac{a^*(1 - \sqrt{a^*})}{C_p \max(n^*, 1)(1 + \sqrt{n^*})} \quad \text{if} \quad 0 < a^* < 1
\]

\[
|k_1|^2 \geq \frac{(1 - 1/\sqrt{a^*})}{C_p \max(n^*, 1)(1 + \sqrt{n^*})} \quad \text{if} \quad 0 < a^* < 1
\]