Sampling Methods for Time Domain Inverse Scattering Problems

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What Are We Looking At?

- Inverse problem: Find geometry of scattering objects from measurements of time-dependent scattered waves associated to several incident waves.
- We will use Sampling Methods to tackle the problem.
What Are Sampling Methods?

- Characteristic feature: Non-iterative methods for support reconstruction
- 1st sampling method was the Linear Sampling Method (LSM) by Colton & Kirsch ’96
- This field was rapidly growing over the last years: Factorization Method (FM) (Kirsch ’98), Needle method (Ikehata ’00), Probe Method (Potthast ’01), No-response test (Potthast & Luke ’03), Scattering Support (Kusiak & Sylvester ’03), ... 
- Features of (LSM) and (FM):
  - Fast imaging techniques, but require multi-static data
  - No need of a-priori information on the nature of the scatterer
Example in the Frequency Domain

Sampling methods as (LSM) or (FM) compute point-wise criterion telling whether a sampling point is inside or outside the scatterer.

**Scatterer**

**Motivation**

Near-field Problems

Near-Field Problems

Conclusion

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**Time Domain Data for Inverse Scattering**

- **Aim:** Sampling methods in the time domain
- **Time-domain data:** causal solutions to the wave equation
- **Hope:** better reconstructions by using many frequencies
- **Natural reconstruction criteria,** no image synthesis necessary
- **Regularization of the method in the time domain**
Related Work (strongly biased!)

- Multi-frequency LSM: Guzina, Cakoni & Bellis ’10
- Sampling methods in the time domain
  - Potthast & Luke ’06
  - Burkhard & Potthast ’09
- Time reversal & Boundary control
  - Blagovescenski ’66; Bingham ’05
  - Bingham, Kurylev, Lassas & Siltanen ’08
  - Dahl, Kirpichnikova & Lassas ’09
  - Oksanen ’11
- Imaging in heterogeneous (cluttered) media: Borcea, Papanicolaou & Tsogka (see Chrysoula’s talk tomorrow)
Overview

1. Motivation, Setting, Related Works

2. Near-Field Inverse Scattering (Chen, Haddar, L & Monk ’10)

3. Far-Field Inverse Scattering (Haddar & L, preprint)

4. Conclusion
The Mathematical Model

- 3D-fundamental solution for
  \[ \partial_{tt} u - \Delta u = 0: \quad k(x, t) = \frac{\delta(t - |x|)}{4\pi |x|} \]

- \( \chi \in C_0^\infty(\mathbb{R}) \) such that
  \( \chi(t) = 0 \) for \( t < 0 \)

- **Incident wave**: point source at
  \( x_0 \in \Gamma \):
  \[ u^i(x, t; x_0) = \chi \ast k(x - x_0, \cdot) = \frac{\chi(t - |x - x_0|)}{4\pi |x - x_0|} \]

- **Scattered field** \( u^s(x, t; x_0) \) solves
  \[ \partial_t^2 u^s - \Delta u^s = 0 \quad (\Omega \times \mathbb{R}), \]
  \[ u^s = -u^i(\cdot, \cdot; x_0) \quad (\partial D \times \mathbb{R}), \quad u^s = 0 \quad (\Omega \times (-\infty, 0)) \]
An Inverse Problem for Near-Field Data

- **Inverse Problem**: Given \( \{ u^s(x, t; x_0) \} \) for all \( x, x_0 \in \Gamma \) and \( t \in \mathbb{R} \), find \( D \)!
- **Linear Sampling Method** uses the near-field operator

\[
(N \phi)(x, t) = \int_{\mathbb{R}} \int_{\Gamma} u^s(x, t - t_0; x_0) \phi(x_0, t_0) \, ds(x_0) \, dt_0
\]

- Roughly speaking: Check whether a point \( z \) is inside \( D \) by checking whether \( u^i(\cdot, \cdot; z) \) belongs to range of \( N \)
- Plot \( z \mapsto 1/\|g_z\| \) where \( Ng_z \approx u^i(\cdot, \cdot; z) \)
Numerical Results

(a) wave speed = 1, source $\sim \sin(4t)e^{-1.6(t-3)^2}$, $\lambda_c = 2\pi/4 \approx 1.6$, 1% added random noise. (b) Frequency domain reconstruction at central wave length $\lambda_c$ using standard frequency domain linear sampling method.
The method can tackle mixed (and unknown!) boundary conditions. (Neumann bc on the “L” – Dirichlet bc on the “I”)

![Diagram of a 2D field with a color scale indicating values from 0.1 to 0.6. The diagram shows a region with different color intensities, indicating variations in the field values.]
Technical Tools

- **Retarded layer potentials**

\[
(SL_{\partial D}\phi)(x, t) = \int_{\partial D} \frac{\phi(x_0, t - |x - x_0|)}{4\pi|x - x_0|} \, ds(x_0) =: u(x, t), \ x \in \Omega
\]

- **Retarded single layer operator on \(\partial D\)**

\[
(S_{\partial D}\phi)(x, t) = \int_{\partial D} \frac{\phi(x_0, t - |x - x_0|)}{4\pi|x - x_0|} \, ds(x_0), \ x \in \partial D
\]

- \(u\) solves \(\partial_t^2u - \Delta u = 0\) in \(\Omega \times \mathbb{R}\) and \(u = S\phi\) on \(\partial D \times \mathbb{R}\)

- Boundary integral equations (BIE): \(u = SL_{\partial D}\phi\) solves wave equation in \(\Omega\) and \(u|_{\partial D \times \mathbb{R}} = -u^i\) iff \(\phi\) solves \(S_{\partial D}\phi = -u^i\)

- \(u\) is causal if \(\phi\) is causal
Function Spaces

- Analysis of retarded BIE via (Fourier-)Laplace transform (Bamberger & Ha Duong '86). **Idea:** Exploit coercivity at complex frequencies $s$ with $\text{Im}(s) = \sigma > 0$.

- Resulting function spaces:

$$W^m_\sigma(\mathbb{R}, X) := \left\{ f \in \mathcal{D}'(\mathbb{R}; X), \ e^{-\sigma t}f \in \mathcal{S}'(\mathbb{R}; X), \right.$$\[\int_{\text{Im}(s) = \sigma} \left| s \right|^{2m} \left\| \mathcal{L}[f](s) \right\|_X^2 \ ds < \infty \}

$$S_{\partial D} : \ W^p_\sigma(\mathbb{R}; H^{-1/2}(\partial D)) \rightarrow W^{p-1}_\sigma(\mathbb{R}; H^{1/2}(\partial D))$$

$$S^{-1}_{\partial D} : \ W^p_\sigma(\mathbb{R}; H^{1/2}(\partial D)) \rightarrow W^{p-2}_\sigma(\mathbb{R}; H^{-1/2}(\partial D)) \quad (!)$$

$$- \int_{\mathbb{R}} e^{-2\sigma t} \int_{\partial D} S^{-1}(\psi) \partial_t \psi \ ds \ dt \geq C \left\| \psi \right\|_{L^2_\sigma(\mathbb{R}; H^{1/2}(\partial D))}^2 \geq 0$$
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Why to Look at Far-Field Data?

- Suitable model if one measures several wavelengths $\lambda_{\text{max}}$ away from the object
- Far-field data often contains more mathematical structure
- There is a richer (better?) theory than in the near-field regime (factorization method)
- Recall what is necessary for a factorization method:
  - Self-adjoint factorization $F = H^* TH$
  - Characterization of the obstacle: $z \in D \iff \phi_z \in \text{Rg } (H^*)$
    (usually easy)
  - “Positivity” of $T$ (sometimes difficult; usually restrictive)
- Method relies on “square root” of $F \Rightarrow$ time domain version of the method is not relied to the frequency version in a simple way
Far-Field Data – Setting

- Incident wave front
  \[ u^i(x, t; \theta) = \chi(t - \theta \cdot x) \]
- Set \( T = \sup_{x \in D} |x| \)
- Scattered field \( u^s(\cdot, \cdot; \theta) \)

\[
\partial_t^2 u^s - \Delta u^s = 0 \quad (\Omega \times \mathbb{R}),
\]

\[
u^s = -u^i(\cdot, \cdot; \theta) \quad (\partial D \times \mathbb{R}),
\]

\[
u^s = 0 \quad (\Omega \times (-\infty, -T))
\]

- Far-field: \( u^\infty(\xi, t) = \lim_{r \to \infty} r u^s(r\xi, t + r) \) for \( \xi \in S^2, \ t \in \mathbb{R} \)

\[
u^s(r\xi, t + r) = u^\infty(\xi, t)/r + O(1/r^2) \text{ as } r \to \infty
\]
Formal Fourier Check-Up for Far-Fields

- Fourier transform: \( \hat{f}(k) = \int_{-\infty}^{\infty} \exp(ikt)f(t) \, dt \) (for real \( k \))
- \( ru^s(r\xi, t + r) \approx u^\infty(\xi, t) \) (for large \( r \))

\[ e^{-ikr} ru^s(r\xi, k) \approx \hat{u}^\infty(\xi, k) \Rightarrow \hat{u}^s(r\xi, k) \approx \frac{\exp(ikr)}{r} \hat{u}^\infty(\xi, k) \]
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e^{-ikr}ru^s(r\xi, k) \approx \hat{u}^\infty(\xi, k) \Rightarrow \hat{u}^s(r\xi, k) \approx \frac{\exp(ikr)}{r} \hat{u}^\infty(\xi, k)
\]

- Far-fields of single-layer potentials:

\[
v(x, t) := (SL\partial_D\phi)(x, t) = \int_{\partial D} \frac{\phi(x_0, t - |x - x_0|)}{4\pi|\!\!|x - x_0|\!\!|} ds(x_0) \Rightarrow v^\infty(\xi, t) = \lim_{r \rightarrow \infty} rv(r\xi, r + t) = \frac{1}{4\pi} \int_{\partial D} \phi(x_0, t + \xi \cdot x_0) ds(x_0)
\]

Hence \( \hat{v}^\infty(\xi, k) = \frac{1}{4\pi} \int_{\partial D} \exp(-ik\xi \cdot x_0)\hat{\phi}(x_0, k) ds(x_0) \)
Inverse Problem – Mathematical Setting

- Incident Herglotz waves:
  For $g : S^2 \times \mathbb{R} \to \mathbb{R}$,
  \[ v_g(x, t) = \int_{S^2} g(\theta, t - \theta \cdot x) \, ds(\theta) \]

- Far-field operator:
  \[ F : g \mapsto u^\infty \] (far-field of scattered field $u^s$ to $v_g$)

- Inverse Problem: Given $F$, find $D$!
- Formally, $F$ can be written as an integral operator
  \[ (Fg)(\xi, t) = \int_{\mathbb{R}} \int_{S^2} v^\infty(\xi, t - t_0; \theta) g(\theta, t_0) \, ds(\theta) \, dt_0, \]
  where $v^\infty(\xi, t; \theta)$ is far-field for incident “wave” $\delta(t - \theta \cdot x)$
Factorization

- **Herglotz operator:** \( Hg = v_g |_{\partial D \times \mathbb{R}} \) where

  \[
v_g(x, t) = \int_{S^2} g(\theta, t - \theta \cdot x) \mathrm{d}s(\theta)
  \]

- Scattered field via boundary integral equations:
  \( u^s = SL_{\partial D}(\phi) \), where \( \phi \) solves a retarded BIE: \( S_{\partial D} \phi = -Hg \)

- Far-field mapping \( R \):

  \[
  R(\phi)(\xi, t) = \frac{1}{4\pi} \int_{\partial D} \phi(x_0, t + \xi \cdot x_0) \mathrm{d}s(x_0)
  \]

- Far-field \( u^\infty \) of \( u^s = SL(\phi) \) is \( R\phi \Rightarrow u^\infty = R\phi = -RS_{\partial D}^{-1}Hg \)

- We are lucky: \( R = H^* \) (for inner product of \( L^2(S^2 \times \mathbb{R}) \))

- Factorization:\( F = -H^* S_{\partial D}^{-1} H \)
Factorization

- Herglotz operator: $Hg = v_g|_{\partial D \times \mathbb{R}}$ where

$$v_g(x, t) = \int_{S^2} g(\theta, t - \theta \cdot x) \, ds(\theta)$$

- Scattered field via boundary integral equations:
  $u^s = SL_{\partial D}(\phi)$, where $\phi$ solves a retarded BIE: $\phi = -S_{\partial D}^{-1}Hg$

- Far-field mapping $R$:
  $$R(\phi)(\xi, t) = \frac{1}{4\pi} \int_{\partial D} \phi(x_0, t + \xi \cdot x_0) \, ds(x_0)$$

- Far-field $u^\infty$ of $u^s = SL(\phi)$ is $R\phi \Rightarrow u^\infty = R\phi = -RS_{\partial D}^{-1}Hg$

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- Factorization: $F = -H^* S_{\partial D}^{-1} H$
Factorization (cont’d)

\[ F = -H^* S_{\partial D}^{-1} H \]
Function Spaces

- Exponentially weighted spaces ⇒ $F$ is well-defined
- Problem: $R = H^*$ only for inner product of $L^2(S^2 \times \mathbb{R})$ – we are doomed to work in Gelfand triples over $L^2$ . . .

From Bamberger & Ha Duong ’86:

$$- \int_{\mathbb{R}} e^{-2\sigma t} \int_{\partial D} S_{\partial D}^{-1}(\psi) \partial_t \psi \, ds(x) \, dt \geq C \| \psi \|^2_{L^2_\sigma(\mathbb{R}; H^{1/2}(\partial D))} \geq 0$$
Function Spaces

- Exponentially weighted spaces \( \Rightarrow F \) is well-defined
- Problem: \( R = H^* \) only for inner product of \( L^2(S^2 \times \mathbb{R}) \) – we are doomed to work in Gelfand triples over \( L^2 \) . . .
- From Bamberger & Ha Duong ’86:

\[
- \int_{\mathbb{R}} \int_{\partial D} S^{-1}_{\partial D}(\psi) \partial_t \psi \, ds(x) \, dt \geq 0 \quad \text{for} \quad \psi \in C_0^\infty(S^2 \times \mathbb{R})
\]
Function Spaces

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$$\int_{\mathbb{R}} \int_{\partial D} \partial_t S_{\partial D}^{-1}(\psi) \psi \, ds(x) \, dt \geq 0 \text{ for } \psi \in C_0^\infty(S^2 \times \mathbb{R})$$

- We are again lucky: $\partial_t F = -H^* \partial_t S_{\partial D}^{-1} H$
- Seek for $X \subset L^2(\mathbb{R}; L^2(S^2))$ and $Y \subset L^2(\mathbb{R}; H^{1/2}(\partial D))$:
  - $H : X \rightarrow Y$ bounded
  - $\partial_t S_{\partial D}^{-1} : Y \rightarrow Y^*$ bounded (and non-negative)
  - Then $F : X \rightarrow X^*$ is bounded, too

- Our choice:
  $X = \{ f \in \mathcal{D}'(\mathbb{R}; L^2(S^2)), (1 + t^2)^{1/2}f \in H^4(\mathbb{R}; L^2(S^2)) \}$
  $Y = \{ f \in \mathcal{D}'(\mathbb{R}; H^{1/2}(\partial D)), (1 + t^2)^{1/2}f \in H^3(\mathbb{R}; H^{1/2}(\partial D)) \}$
Characterization of $D$

- $-\partial_t F : X \rightarrow X^*$ and $\partial_t S_{\partial D}^{-1} : Y \rightarrow Y^*$ non-negative $\Rightarrow$
  - Existence of square roots $-\partial_t F = Q^*_F Q_F$ and $\partial_t S_{\partial D}^{-1} = Q^*_S Q_S$

- Two factorizations:
  
  $$-\partial_t F = Q^*_F Q_F = (Q_S H)^*(Q_S H)$$

- In this situation: $\text{Rg} \left( Q^*_F \right) = \text{Rg} \left( H^* Q^*_S \right)$

- The far-field of a point source at $z$,
  
  $$\phi_z(x, t) = \frac{\chi(t - |x - z|)}{|x - z|}, \quad x \neq z,$$

  equals $\phi^\infty_z(\xi, t) = \chi(t + \xi \cdot z)$

- Range identity yields (skipping technicalities . . . )
  
  If $z \in D$ $\Rightarrow$ $\phi^\infty_z \in \text{Rg} \left( Q^*_F \right)$
  
  If $z \notin \overline{D}$ $\Rightarrow$ $\phi^\infty_z \notin \text{Rg} \left( Q^*_F \right)$
Conclusion

- Inverse scattering in the time domain
- Linear sampling method in the time domain for near-field data
- Factorization method in the time domain for far-field data
- Explicit Characterization of Dirichlet objects from far-field measurements
- Analysis uses Fourier transform – the characterization does not

Thanks for your attention!