Enhancement of Near Cloaking Using GPT Vanishing Structures

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This talk is based on a joint work with

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Outline

- Cloaking and near cloaking
- Generalized Polarization Tensors
- GPT and imaging
- GPT and near cloaking
- GPT vanishing structures
Transformation of PDE

- Let $\Lambda[\sigma]$ be the Dirichlet-to-Neumann map corresponding to the conductivity distribution $\sigma$, i.e.,

$$\Lambda[\sigma](\phi) = \sigma \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega}$$

where $u$ is the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0, & \text{in } \Omega, \\ u = \phi, & \text{on } \partial \Omega. \end{cases}$$

- If $F$ is a diffeomorphism of $\Omega$ which is identity on $\partial \Omega$, then

$$\Lambda[\sigma] = \Lambda[F_* \sigma]$$

where $F_* \sigma$ is the push-forward of $\sigma$ by $F$:

$$F_* \sigma(y) = \frac{DF(x)\sigma(x)DF(x)^t}{\det(DF(x))}, \quad x = F^{-1}(y).$$
Singular transformation

- Define $F : \{x : 0 < |x| < 2\} \rightarrow \{x : 1 < |x| < 2\}$ by
  
  $$F(x) := \left(1 + \frac{|x|}{2}\right) \frac{x}{|x|}.$$ 

- Then, $\Lambda[1] = \Lambda[F_1].$
- Things inside $\{|x| < 1\}$ are cloaked by the DtN map (Greenleaf-Lassas-Uhlmann (2003)).
- $F_1$ is singular on $|x| = 1$ (0 in the normal direction, $\infty$ in tangential direction, 2D)
Near cloaking

Blowing-up a small ball (Kohn-Shen-Vogelius-Weinstein (2008))

• For a small number $\rho$, let

$$
\sigma = \begin{cases} 
\gamma & \text{if } |x| < \rho, \\
1 & \text{if } \rho \leq |x| \leq 2.
\end{cases}
$$

(\(\gamma\) can be 0 (the core is insulated) or \(\infty\) (perfect conductor))

• Let

$$
F(x) = \begin{cases} 
\left(\frac{2 - 2\rho}{2 - \rho} + \frac{1}{2 - \rho}|x|\right) \frac{x}{|x|} & \text{if } \rho \leq |x| \leq 2, \\
\frac{x}{\rho} & \text{if } |x| \leq \rho.
\end{cases}
$$

Then $F$ maps $B_2$ onto $B_2$ and blows up $B_\rho$ onto $B_1$. 
• Then,

\[ \| \Lambda[F*\sigma] - \Lambda[1] \| \leq C\rho^2. \]

• Further development toward acoustic cloaking:
  Kohn-Onofrei-Vogelius-Weinstein, Liu, Nguyen.

• Approximate cloaking by truncation: Greenleaf \textit{et al.}
Enhancement of near cloaking

**Main result:** If we coat the small disk of radius \( \rho \) by multi-layered structure whose **generalized polarization tensors** up to order \( N \) vanish (GPT vanishing structure of order \( N \)), then the cloaking effect is enhanced to \( C(2\rho)^{2N+2} \) (\( C \) independent of \( \rho \) and \( N \)).

**Remark:** \( \rho < \frac{1}{2} \) is enough!
Figure: Near cloaking & an enhancement
Neutral inclusion of Hashine: GPT vanishing structure of order 1
Generalized Polarization Tensors

Conductivity distribution:

\[ \sigma = \chi(\mathbb{R}^d \setminus \Omega) + \gamma \chi(\Omega). \]

Suppose \( \Omega \) is a single inclusion or multiple inclusions and consider

\[
\begin{aligned}
\nabla \cdot \sigma \nabla u &= 0 \quad \text{in } \mathbb{R}^d, \\
\n\nabla \cdot \sigma \nabla u &= 0 \quad \text{in } \mathbb{R}^d, \\
\n&\text{as } |x| \to \infty.
\end{aligned}
\]

The dipolar asymptotic expansion at infinity:

\[
u(x) = a \cdot x - \frac{1}{\omega_d} \frac{\langle a, Mx \rangle}{|x|^d} + O(|x|^{-d}), \quad \text{as } |x| \to \infty.
\]

\[ M = M(\gamma, \Omega) = (m_{ij}) : \text{the Polarization Tensor associated with } \Omega \text{ (or more precisely } \sigma). \]
For a given entire harmonic function $H$, consider

\[
\begin{cases}
\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \mathbb{R}^d, \\
u(x) - H(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty.
\end{cases}
\]

**Multipolar expansions:**

\[
u(x) = H(x) + \sum_\alpha \sum_\beta \frac{(-1)^{|eta|}}{\alpha! \beta!} \partial^\alpha H(0) \partial^\beta \Gamma(x) m_{\alpha \beta}, \quad |x| \to \infty.
\]

\{m_{\alpha \beta}\} : Generalized Polarization Tensors (GPT).

($\Gamma(x)$: the fundamental solution for the Laplacian.)
Equivalent ellipse

If \( \gamma \) is constant, then there is a canonical correspondence between the class of ellipses (ellipsoids) and the class of PTs:

If \( B \) is an ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \), then

\[
M(\gamma, B) = (\gamma - 1)|B| \begin{bmatrix}
\frac{a + b}{a + \gamma b} & 0 \\
0 & \frac{a + b}{b + \gamma a}
\end{bmatrix}.
\]
Equivalent ellipse (ellipsoid) = ellipse with the same PT:
Imaging by GPT (by Ammari-K-Lim-Zribi)

- All GPTs determine the shape and the conductivity uniquely (Ammari-K 03)
- PT of order 1 yields the equivalent ellipse.

**Aim:** Make use of \[ \sum_{|\alpha|+|\beta|\leq K} a_\alpha b_\beta m_{\alpha\beta} \] for a fixed \( K \geq 2 \) to image finer details of the shape of the inclusion.

**Optimization Problem:** Let \( \Omega \) be the target domain. Minimize over \( D \)

\[
J[D] := \frac{1}{2} \sum_{|\alpha|+|\beta|\leq K} w_{|\alpha|+|\beta|} \left| \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D) - \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, \Omega) \right|^2.
\]

- \( w_{|\alpha|+|\beta|} \) are binary weights: \( w_{|\alpha|+|\beta|} = 1 \) (on) or 0 (off).
- A good choice for the initial guess: the equivalent ellipse.
An asymptotic expansion of GPT due to small boundary change:

- \( \varepsilon \)-perturbation of \( D \):

\[
\partial D_\varepsilon := \{ \tilde{x} = x + \varepsilon h(x)\nu(x) \mid x \in \partial D \}.
\]

- Asymptotic formula: Suppose that \( H = \sum_{\alpha} a_\alpha x^\alpha \) and \( F = \sum_{\beta} b_\beta x^\beta \) are harmonic polynomials. Then

\[
\sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D_\varepsilon) - \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D) = \varepsilon(k - 1) \int_{\partial D} h \left[ \frac{\partial v}{\partial \nu} \bigg|_{-} - \frac{\partial u}{\partial \nu} \bigg|_{-} + \frac{1}{k} \frac{\partial u}{\partial T} \bigg|_{-} - \frac{\partial v}{\partial T} \bigg|_{-} \right] d\sigma + O(\varepsilon^2),
\]
where

\[
\begin{align*}
\Delta u &= 0, & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\
u_+ \mid - u_- \mid &= 0, & \text{on } \partial D, \\
\frac{\partial u}{\partial \nu} \mid - k \frac{\partial u}{\partial \nu} \mid &= 0, & \text{on } \partial D, \\
(u - H)(x) &= O(|x|^{-1}) & \text{as } |x| \to \infty,
\end{align*}
\]

and

\[
\begin{align*}
\Delta v &= 0, & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\
v \mid - v_- \mid &= 0, & \text{on } \partial D, \\
\frac{\partial v}{\partial \nu} \mid \mid - \frac{\partial v}{\partial \nu} \mid &= 0, & \text{on } \partial D, \\
(v - F)(x) &= O(|x|^{-1}) & \text{as } |x| \to \infty.
\end{align*}
\]
• Shape derivative: Let

\[ \phi_{HF}^D (x) = (k - 1) \left[ \frac{\partial v}{\partial \nu} \bigg|_{\partial D} \frac{\partial u}{\partial \nu} \bigg|_{\partial D} + \frac{1}{k} \frac{\partial u}{\partial T} \bigg|_{\partial D} \frac{\partial v}{\partial T} \bigg|_{\partial D} \right], \]

and

\[ \delta_{HF}^D = \sum_{\alpha, \beta} a_\alpha b_\beta m_{\alpha \beta} (k, D) - \sum_{\alpha, \beta} a_\alpha b_\beta m_{\alpha \beta} (k, B). \]

Then

\[ \langle dS J[D], h \rangle_{L^2(\partial D)} = \sum_{|\alpha| + |\beta| \leq K} w_{|\alpha| + |\beta|} \delta_{HF}^D \langle \phi_{HF}^D, h \rangle_{L^2(\partial D)}. \]

• Determination of location: Let \( Ga_l := e_l \) and \( Gb_l := 2e_l \), \( l = 1, \ldots, d \). Then \( \frac{m_{\alpha l} \beta_l}{m_{\alpha l} \beta_l} \) is the center of mass of \( B \) if \( B \) is a ball.
Figure: $K = 6$, 6 iterations.
Figure: Reconstruction of clusters of inclusions. The upper images: the equivalent ellipses, and the lower ones: results after 6 iterations.
Figure: After 6 iterations
GPT carry information on topology!!! (recent computation by Yu).
Back to the near-cloaking

• $\Lambda[F_\sigma] = \Lambda[\sigma]$ and

$$\Lambda[\sigma](\phi)(x) = \Lambda[1](\phi)(x) + \nabla U(0) \cdot M \frac{\partial}{\partial \nu_x} \nabla_y G(x, 0) + \text{h.o.t.}, \quad x \in \partial \Omega,$$

where $U$ is the solution to

$$\begin{cases}
\Delta U = 0 & \text{in } \Omega, \\
U = \phi & \text{on } \partial \Omega,
\end{cases}$$

$M$ is the polarization tensor of $B_\rho$, and $G(x, y)$ is the Green function for $\Omega$. (The expansion holds uniformly for $\gamma$).

• PT for $B_\rho$ (with conductivity $\gamma$): $M = \frac{2(\gamma - 1)}{\gamma + 1} |B_\rho| l$.

• Thus,

$$\|\Lambda[F_\sigma] - \Lambda[1]\| \leq C \rho^2.$$
Small volume expansion

• Let $u$ be the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 \text{ in } \mathbb{R}^2, \\ u(x) - H(x) = O(|x|^{-1}) \text{ as } |x| \to \infty. \end{cases}$$

• Then, as $|x| \to \infty$,

$$(u - H)(x) = -\sum_{m,n=1}^{\infty} \left[ \frac{\cos m\theta}{2\pi mr^m} \left( M_{mn}^{cc} a_n^c + M_{mn}^{cs} a_n^s \right) + \frac{\sin m\theta}{2\pi mr^m} \left( M_{mn}^{sc} a_n^c + M_{mn}^{ss} a_n^s \right) \right]$$

where $H(x) = H(0) + \sum_{n=1}^{\infty} r^n (a_n^c \cos n\theta + a_n^s \sin n\theta)$.

• $M_{mn}^{cc}$, $M_{mn}^{cs}$, $M_{mn}^{sc}$, $M_{mn}^{ss}$ are called (condensed) GPT.
If γ (and hence σ) is radial, then

- Because of the symmetry of the disc,

\[ M^{cs}_{mn}[\sigma] = M^{sc}_{mn}[\sigma] = 0 \quad \text{for all } m, n, \]
\[ M^{cc}_{mn}[\sigma] = M^{ss}_{mn}[\sigma] = 0 \quad \text{if } m \neq n, \]

and

\[ M^{cc}_{nn}[\sigma] = M^{ss}_{nn}[\sigma] \quad \text{for all } n. \]

- Let \( M_n = M^{cc}_{nn}, \ n = 1, 2, \ldots \). Then, as \(|x| \to \infty|\),

\[ (u - H)(x) = -\sum_{n=1}^{\infty} \left[ \frac{M_n}{2\pi nr^n} (a^c_n \cos n\theta + a^s_n \sin n\theta) \right] \]

where \( H(x) = H(0) + \sum_{n=1}^{\infty} r^n (a^c_n \cos n\theta + a^s_n \sin n\theta). \)
Two important lemmas: Let

$$\sigma = \begin{cases} 
\gamma_0 \text{ (const)}, & |x| < 1, \\
\gamma & 1 \leq |x| < 2, \\
1 & 2 \leq |x|. 
\end{cases}$$

where $\gamma$ is radial.

• Then

$$\left( \Lambda[\sigma \left( \frac{1}{\rho} x \right)] - \Lambda[1] \right)(f) = \sum_{k=-\infty}^{\infty} \frac{2|k|\rho^2|k|M_{|k|}[\sigma]}{2\pi |k| - \rho^2|k|M_{|k|}[\sigma]} f_k e^{ik\theta}.$$ 

• $|M_k[\sigma]| \leq 2\pi k 2^{2k}$ for all $k$. 
Enhancement of near cloaking

**GPT vanishing structure of order** \( N \): \( M_k = 0 \) for \( k \leq N \).

Let \( \gamma \) be a GPT vanishing structure of order \( N \).

- Let \( \sigma^N(x) = \sigma(\frac{1}{\rho}x) \). Then

  \[
  \left( \Lambda[\sigma^N] - \Lambda[1] \right)(f) = \sum_{|k| > N} \frac{2|k|\rho^{2|k|}M_{|k|}[\sigma]}{2\pi|k| - \rho^{2|k|}M_{|k|}[\sigma]} f_k e^{ik\theta}.
  \]

- Using the transformation blowing up a small ball, we can get a near-cloaking structure such that

  \[
  \| \Lambda[\sigma^N] - \Lambda[1] \| = \| \Lambda[F_\ast \sigma^N] - \Lambda[1] \| \leq C(2\rho)^{2N+2}.
  \]
Multi-layered structure

- For a positive integer $N$, let $1 = r_{N+1} < r_N < \ldots < r_1 = 2$ and define
  $$A_j := \{r_{j+1} < r \leq r_j\}, \quad j = 1, 2, \ldots, N.$$  

- $A_0 = \mathbb{R}^2 \setminus B_2$, $A_{N+1} = B_1$.

- Set $\sigma_j$ to be the conductivity of $A_j$ for $j = 1, 2, \ldots, N + 1$, and $\sigma_0 = 1$. Let
  $$\sigma = \chi(A_0) + \sum_{j=1}^{N} \sigma_j \chi(A_j) + \sigma_{N+1} \chi(A_{N+1}).$$

($\sigma_{N+1}$ may (or may not) be fixed: $\sigma_{N+1}$ is fixed to be 0 if the core is insulated.)
• To compute $M_k$, we look for solutions $u_k$ to

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \mathbb{R}^2$$

of the form

$$u_k(x) = a_j^{(k)} r^k \cos k\theta + \frac{b_j^{(k)}}{r^k} \cos k\theta \quad \text{in } A_j, \quad j = 0, 1, \ldots, N + 1,$$

with $a_0^{(k)} = 1$ and $b_{N+1}^{(k)} = 0$.

• Then $u_k$ satisfies

$$(u_k - H)(x) = \frac{b_0^{(k)}}{r^k} \cos k\theta \quad \text{as } |x| \to \infty.$$

with $H(x) = r^k \cos k\theta$.

• Hence, $M_k = -2\pi k b_0^{(k)}$. 
• The transmission conditions on the interface \( \{ r = r_j \} \):

\[
\begin{bmatrix}
a_{j}^{(k)} \\
b_{j}^{(k)}
\end{bmatrix} = \frac{1}{2\sigma_j} \begin{bmatrix}
\sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1})r_j^{-2k} \\
(\sigma_j - \sigma_{j-1})r_j^{2k} & \sigma_j + \sigma_{j-1}
\end{bmatrix} \begin{bmatrix}
a_{j-1}^{(k)} \\
b_{j-1}^{(k)}
\end{bmatrix},
\]

and hence

\[
\begin{bmatrix}
a_{N+1}^{(k)} \\
0
\end{bmatrix} = \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix}
\sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1})r_j^{-2k} \\
(\sigma_j - \sigma_{j-1})r_j^{2k} & \sigma_j + \sigma_{j-1}
\end{bmatrix} \begin{bmatrix}
1 \\
b_{0}^{(k)}
\end{bmatrix}.
\]

• Let

\[
P^{(k)} = \begin{bmatrix}
p_{11}^{(k)} & p_{12}^{(k)} \\
p_{21}^{(k)} & p_{22}^{(k)}
\end{bmatrix} := \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix}
\sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1})r_j^{-2k} \\
(\sigma_j - \sigma_{j-1})r_j^{2k} & \sigma_j + \sigma_{j-1}
\end{bmatrix}.
\]

Then,

\[
M_k = -2b_{0}^{(k)} = \frac{2p_{21}^{(k)}}{p_{22}^{(k)}}.
\]
GPT vanishing structure

- GPT vanishing structure of order $N$: $M_k = 0$ for $k = 1, \ldots, N$, or $p_{21}^{(k)} = 0$, $k = 1, \ldots, N$,

where

$$
\begin{bmatrix}
  p_{11}^{(k)} & p_{12}^{(k)} \\
p_{21}^{(k)} & p_{22}^{(k)}
\end{bmatrix} := \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix}
  \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1})r_j^{-2k} \\
(\sigma_j - \sigma_{j-1})r_j^{2k} & \sigma_j + \sigma_{j-1}
\end{bmatrix}.
$$

- Solve the equations for $r_N < \ldots < r_2$ and $\sigma_N, \ldots, \sigma_1$.
- If $N = 1$, $(\sigma_2 - \sigma_1)(\sigma_1 + 1)r_2^2 + (\sigma_2 + \sigma_1)(\sigma_1 - 1) = 0$.
- For $N = 2, 3, \ldots$, can be solved by hand.
- For arbitrary $N$, the equation is a non-linear algebraic equation and existence of solution is yet to be proved.
$M = 0$ (GPT vanishing structure of order 1) : the neutral inclusion of Hashine
GPT vanishing structure

Figure: The conductivity of the core is not fixed. \( N = 3 \)
Figure: The conductivity of the core is not fixed. $N = 6$
Figure: The conductivity of the core is not fixed. $N = 9$
Figure: The conductivity of the core is fixed to be 0. $N = 3$
Figure: The conductivity of the core is fixed to be 0. \( N = 6 \)
Conclusion

• By using the GPT vanishing structures, we improve near cloaking effect to \((2\rho)^{2N+2}\).

• If we take \(N\) large, then \(\rho < \frac{1}{2}\) is enough to have a good cloaking effect. *(No need of the transformation!)*

• Problems:
  
  • Existence of GPT vanishing structures
  • What is the limit of the GPT vanishing structures when \(N \to \infty\)?

• Coming attractions:
  
  • Numerical simulations for the enhancement of the near-cloaking
  • Extension to the Helmholtz equation.
  • Analysis of cloaking by the anomalous localized resonance.
Thank you!