Medical Imaging using Computational Conformal/Quasi-conformal Geometry

Part I of the sequel of 2 talks.
More applications of QC theory will be presented by Ronald Lui

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Outline Of The Talk

- **Motivation**
- Part I: Conformal Geometry & Applications
- Part II: Quasi-conformal Geometry & Applications
- Conclusion
**What is Medical Morphometry?**

**Medical Morphometry**: Tracking of shape changes/abnormality; analysis of medical images.

**Main goal**: Generate diagnostic images for visualization of structural changes.

**How Conformal/Quasi-conformal theory help?**
Brain Mapping Tasks

Automatic identification and localization of structures and function

Statistical shape analysis

Spatial normalization in Canonical Space

PDEs, ...

Shape representation
Hippocampus: long-term memory & spatial navigation; Shape analysis for Alzheimer's disease

Lateral ventricles: fluid-filled structures deep in the brain; Enlarged in disease; Shape analysis for measures of disease progression.
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- **Part I: Conformal Geometry & Applications**
  - Basic Mathematical Background
  - Computational Algorithms
  - Applications
- **Part II: Quasi-conformal Geometry & Applications**
- Conclusion
What is Conformal map?

- Conformal map \( f : M \rightarrow N \) = preserves inner product up to a scaling factor (the conformal factor \( \lambda \)).

- Mathematically, \( f^*(ds_N^2) = \lambda(x_1, x_2)ds_M^2 \) where \( ds_M^2 = \sum_{i,j=1}^{2} g_{ij}dx^i \wedge dx^j \)
Why Conformal for Brain Mapping?

- Metric preserved up to scaling $\iff$ Local geometry preserved!
- Angle-preserving $\iff$ inherits a natural orthogonal grid on the surface.
- Simple $(g_{ij})$ Matrix $\iff$ simple differential operator expression on the parameter domain and simple projected equations.

\[ \Phi \]
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- Conclusion
Computation of Conformal Maps

- Genus zero surface conformal parameterization

**Theorem: (Genus 0)**

\[
\begin{align*}
    f : S_1 & \rightarrow S_2 \\
    \text{Genus 0} & \\
    \text{Conformal}
\end{align*}
\]

**Minimize:**

\[
E_{\text{harmonic}}(f) = \frac{1}{2} \int_{S_1} |\nabla_{g_1} f|_{g_2}^2 dS_1
\]

\[
|\nabla_{g_1} f|_{g_2}^2 = tr(\nabla^T f \nabla f)
\]

Discrete version of Harmonic energy:

\[
E(f) = \langle f, f \rangle = \frac{1}{2} \sum_{[u,v] \in K} k_{u,v} |f(u) - f(v)|^2
\]  

\[
\Delta_{PL}(f) = \sum_{[u,v] \in K} k_{u,v} (f(u) - f(v))
\]

(Harmonic Energy)  

(Discrete Laplacian)
Genus-0 Conformal Maps: Examples

Two brain surfaces are of the same subject at different time.

Conformal mapping:

Good parameter domain
Curvature flow method

Basic idea: conformally deform Riemannian metric to another Riemannian metric to achieve prescribed curvature

Curvature Flow Algorithm:

\[ g = (g_{ij}) \quad \text{(Riemannian metric)} \quad \bar{g} = e^{2u} g \quad \text{(Another Riemannian metric)} \]

The Gaussian curvature \( \bar{K} \) under \( \bar{g} \) is:

\[ \bar{K} = e^{2u} (-\Delta_g u + K) \]

WE HAVE THE CURVATURE FLOW EQUATION:

\[ \frac{dg_{ij}(t)}{dt} = 2(\bar{K} - K)g_{ij}(t) \]

(Flow the metric to a new metric with prescribed curvature \( \bar{K} \))

\[ g(t) = e^{2u(t)} g(0) \quad \rightarrow \quad \frac{du(t)}{dt} = 2(\bar{K} - K) \]

(Yamabe Equation)

(Rewriting the equation in term of \( u(t) \))
Examples: Curvature Flow Method

(5 landmarks)

(7 landmarks)
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Application: Brain Registration

Brain Conformal Parameterization:
A canonical domain for brain surface analysis!

Genus 0

Open surface (disk is removed at the back)
Application: Brain Registration

Conformal Slit Map: (using curvature flow method)
Sulcal landmarks are mapped to circular slits or horizontal slit

Circular Slit map
Horizontal Slit map

Another brain
Solving PDEs on surfaces: Conformal Approach

- Goal: Solve equations on the surface by mapping it onto the 2D conformal parameter domain.
- Differential operators are computed on 2D domain with simple formula.
- Example: \( \frac{\partial \tilde{u}}{\partial t} = -\frac{1}{\lambda} (\tilde{u} \cdot \nabla) \tilde{u} + \frac{\nu}{\lambda} \nabla^2 \tilde{u} + f \) (Navier-Stokes)

<table>
<thead>
<tr>
<th>General parameterization:</th>
<th>Conformal parameterization:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gradient:</strong> ( \nabla_M f = (g^{11}\partial_x f + g^{21}\partial_y f)i + (g^{12}\partial_x f + g^{22}\partial_y f)j )</td>
<td>( \nabla_M f = \frac{1}{\lambda} \partial_x f i + \frac{1}{\lambda} \partial_y f j )</td>
</tr>
<tr>
<td><strong>Laplacian:</strong> ( \Delta_M f = \frac{1}{\sqrt{g^{11}g^{22} - g^{12}g^{12}}} \left[ \partial_x (\sqrt{g^{11}g^{22} - g^{12}g^{12}} (g^{11}\partial_x f + g^{21}\partial_y f)) + \partial_y (\sqrt{g^{11}g^{22} - g^{12}g^{12}} (g^{12}\partial_x f + g^{22}\partial_y f)) \right] )</td>
<td>( \Delta_M f = \frac{1}{\lambda} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) )</td>
</tr>
<tr>
<td><strong>Divergence:</strong> ( \text{div}_M(Xi + Yj) = \frac{1}{\sqrt{g^{11}g^{22} - g^{12}g^{12}}} \left[ \partial_x \left( \sqrt{g^{11}g^{22} - g^{12}g^{12}} X \right) + \partial_y \left( \sqrt{g^{11}g^{22} - g^{12}g^{12}} Y \right) \right] )</td>
<td>( \text{div}_M(Xi + Yj) = \frac{1}{\lambda} \left[ \partial_x (\lambda X) + \partial_y (\lambda Y) \right] )</td>
</tr>
</tbody>
</table>
Applications: Imaging on surfaces

- **Denoising**

- **Inpainting**

\[
E_{TV}^S(u) = \int_S \left[ \|\nabla_S u\|_S + |u - f|^2 \right] dS
\]

\[
\frac{\partial u}{\partial t} = 2\nu(f - u) + \text{div}_M \left( \frac{\nabla_M u}{|\nabla_M u|_M} \right)
\]

\[
= 2\nu(f - \zeta) + \frac{1}{\lambda} \text{div} \left( \sqrt{\lambda} \frac{\nabla \zeta}{|\nabla \zeta|} \right)
\]

\[
\frac{\partial u}{\partial t} = \text{div}_M \left[ \frac{g(\kappa)}{|\nabla_M u|_M} \nabla_M u \right] = \frac{1}{\lambda} \nabla \cdot \left[ \sqrt{\lambda} g(\kappa) \frac{\nabla \zeta}{|\nabla \zeta|} \right], \quad x \in \phi^{-1}(D)
\]

and \( \zeta = \zeta^0, \quad x \in \phi^{-1}(D^c) \)

\[
\kappa = \text{div}_M \left( \frac{\nabla_M u}{|\nabla_M u|_M} \right) = \frac{1}{\lambda} \nabla \cdot \left( \sqrt{\lambda} \frac{\nabla \zeta}{|\nabla \zeta|} \right)
\]

( \( \phi = \) conformal parametrization of \( M \) and \( \zeta = u \circ \phi \) )
Application: Automatic Sulcal Landmark Tracking

Extraction of high mean curvature region by Chan-Vese segmentation

\[ F(c_1, c_2, \psi) = \int_S (I_f - c_1)^2 H(\psi) dS + \int_S (I_f - c_2)^2 (1 - H(\psi)) dS + \nu \int_S |\nabla_S H(\psi)| dS \]

\[ c_1 = \frac{\int_D I_f \circ \phi(x, y) H(\psi \circ \phi(x, y)) \lambda(x, y) dxdy}{\int_D H(\psi \circ \phi(x, y)) \lambda(x, y) dxdy} \]

\[ c_2 = \frac{\int_D I_f \circ \phi(x, y) (1 - H(\psi \circ \phi(x, y))) \lambda(x, y) dxdy}{\int_D (1 - H(\psi \circ \phi(x, y))) \lambda(x, y) dxdy} \]

Euler Lagrange equation is:

\[ \frac{\partial \psi}{\partial t} = \lambda \delta(\psi) \left[ \nu \text{div} \left( \frac{\nabla_S \psi}{||\nabla_S \psi||_S} \right) - (I_f - c_1)^2 - (I_f - c_2)^2 \right] \]

or

\[ \frac{\partial \psi \circ \phi}{\partial t} = \lambda \delta(\psi \circ \phi) \left[ \nu \frac{1}{\lambda} \text{div} \left( \sqrt{\lambda} \frac{\nabla \psi \circ \phi}{||\nabla \psi \circ \phi||} \right) - (I_f \circ \phi - c_1)^2 - (I_f \circ \phi - c_2)^2 \right] \]
Application: Shape Analysis w/ Conformal Structure

AIDS

Healthy (21 yrs old)
Application: Shape Analysis w/ Conformal Structure

AIDS

Healthy (21 yrs old)

Aspect ratio tells the conformal similarity
Application: Shape Analysis w/ Conformal Structure

HEALTHY  Patient (William Syndrome)

Conformal dissimilarity is measured by the locations and radii of circles

Fixing two circles to the center to remove the Mobius ambiguity
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What is Quasi-conformal map?

- Generalization of conformal maps (angle-preserving);
- Orientation preserving homeomorphism between Riemann surfaces;
- Bounded conformality distortion;
- Intuitively, map infinitesimal circle to ellipse;
- Mathematically, it satisfies: \( \frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z} \)

\[
\begin{align*}
\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right); \\
\frac{\partial f}{\partial z} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)
\end{align*}
\]

- Beltrami coefficient:
  - Measure conformality distortion;
  - Invariant under conformal transformation

Beltrami coefficient: \( K = \frac{1 + |\mu|}{1 - |\mu|} \)

\( \arg(\mu)/2 \)

Conformal \( \iff \mu = 0 \iff \frac{\partial f}{\partial \bar{z}} = 0 \)
Example: Quasi-conformal

**Conformal**

\[ f^* (ds_E^2) = \lambda |dz|^2 \]

**Quasi-conformal**

\[ f^* (ds_E^2) = \left| \frac{\partial f}{\partial \bar{z}} \right|^2 |dz + \mu(z)d\bar{z}|^2 \]
Why Quasi-conformal?

1. **Natural deformations** are unlikely to be rigid, or isometric, or even conformal. The search space for the mapping should include all diffeomorphisms.

2. **Quasi-conformal Geometry** studies the deformation pattern between shapes. It effectively measures the conformality distortion under the deformation.
Goal:
Look for a simple representation of surface diffeomorphisms, called Beltrami representation. (Lui & Wong et al. 2009)

Theorem:
Let $S_1$ and $S_2$ be two surfaces. Suppose $f : S_1 \rightarrow S_2$ is a surface diffeomorphism from $S_1$ and $S_2$. Given 3 points correspondence $\{p_1, p_2, p_3 \in S_1\} \leftrightarrow \{f(p_1), f(p_2), f(p_3) \in S_2\}$. $f$ can be represented by a unique Beltrami coefficient $\mu : S_1 \rightarrow \mathbb{C}$.
Motivation

Part I: Conformal Geometry & Applications

Part II: Quasi-conformal Geometry & Applications
  - Basic Mathematical Background
  - Computational Algorithms
  - Applications

Conclusion
Computation of Quasi-conformal map

Given a Beltrami coefficient, how to get the quasi-conformal map?

Answer: 1. Beltrami Holomorphic Flow method;
2. Curvature flow method to convert QC into C
(will be presented by Ronald Lui)

Theorem: \( (Beltrami \ Holomorphic \ Flow \ method) \)

\[
\text{If:} \quad \mu(t)(z) = \mu(z) + t\nu(z) + t\epsilon(t)(z) \\
\text{Then:} \quad f^{\mu(t)}(w) = f^{\mu}(w) + t\hat{f}^{\mu}[\nu](w) + o(t)
\]

where \( \hat{f}[\nu](w) = -\frac{f^{\mu}(w)(f^{\mu}(w) - 1)}{\pi} \int_{\mathbb{C}} \frac{\nu(z)((f^{\mu})_{z}(z))^2}{f^{\mu}(z)(f^{\mu}(z) - 1)(f^{\mu}(z) - f^{\mu}(w))} \, dx \, dy. \)

\[
\mu_0 = 0 \quad \rightarrow \quad \mu_1 = \mu/N \quad \rightarrow \quad \ldots \quad \rightarrow \quad \mu_N = \mu
\]

\[
f^0 = \text{id} \quad \underbrace{\text{BHF}}_{f^{\mu/N}} \quad \underbrace{\text{BHF}}_{\ldots} \quad \underbrace{\text{BHF}}_{f^{\mu}}
\]
QC parameterization 
example of simply – connected domain

\[ \mu = \frac{z - z_0}{2\sqrt{1 + h^2}} \]

QC parameterization 
example of multiply – connected domain

(a) checker-board 
(b) \( \mu(z) = 0 \) 
(c) \( z_0 = 0 \) 
(d) \( z_0 = 0.5 + i0.5h \) 
(e) \( z_0 = 1 + ih \)
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**Quasi-conformal for Registration**

**Example:** *Hippocampal registration with geometric matching*

Hippocampus = limbic system; important role for long-term memory and spatial navigation

- No well-defined landmarks.
- Meaningful surface registration = DIFFICULT!

Optimizing a compounded energy:

\[
E_{shape}(f^\mu) = \alpha \int_D |\mu|^2 + \beta \int_D (H_1 - H_2(f^\mu))^2 + \gamma \int_D (K_1 - K_2(f^\mu))^2
\]

**Theorem:** Let \( S_1 \) and \( S_2 \) be two surfaces (e.g. HP surfaces). Suppose \( f : S_1 \rightarrow S_2 \) is a registration between \( S_1 \) and \( S_2 \). Assume \( \alpha, \beta, \gamma \neq 0 \), then:

\[
E_{shape}(f) = 0 \iff S_1 \text{ is equal to } S_2 \text{ up to a rigid motion}
\]
Quasi-conformal for Registration

Example: Hippocampal registration with geometric matching

**Basic Idea:** Find a registration $f$ that minimizes the following energy functional:

$$E_{\text{shape}}(f^\mu) = \alpha \int_D |\mu|^2 + \beta \int_D (H_1 - H_2(f^\mu))^2 + \gamma \int_D (K_1 - K_2(f^\mu))^2$$

**Optimization:** Minimize with respect to the the Beltrami coefficient

Easily control the diffeomorphic property!

Simplified optimization problem:

$$\min_{f \in \mathbb{F}_{\text{Diff}}} E_0(f) \text{ where } \mathbb{F}_{\text{Diff}} = \{ f : S_1 \to S_2 : f \text{ is a diffeomorphism} \}$$

$$\min_{\mu \in \mathbb{F}_{\text{BC}}} E(\mu) \text{ where } \mathbb{F}_{\text{BC}} = \{ \mu : S_1 \to \mathbb{D} : |\mu|_\infty < 1 \} \text{ is the set of BCs.}$$

Compute the direction of descent and adjust $f$ through adjusting BC.

**Advantage:**
1. BC doesn’t need to be 1-1, onto. Only constraint is norm $< 1$.
2. The constraint can be easily controlled
   (starting from conformal map with BC $= 0$ everywhere)
Quasiconformal for Registration

Energy:

\[ E_{\text{shape}}(\mu) = \alpha \int_D |\mu|^2 + \beta \int_D (H_1 - H_2(f^\mu))^2 + \gamma \int_D (K_1 - K_2(f^\mu))^2 \]

Euler-Lagrange Equation:

\[
\frac{d}{dt} \bigg|_{t=0} E_{\text{shape}}(\mu + tv) = \int_D \frac{d}{dt} \bigg|_{t=0} |\mu + tv|^2 + \int_D \frac{d}{dt} \bigg|_{t=0} (H_1 - H_2(f^{\mu+tv}))^2 + \int_D \frac{d}{dt} \bigg|_{t=0} (K_1 - K_2(f^{\mu+tv}))^2 \\
= 2 \int_D \mu \cdot v - 2 \int_D (H_1 - H_2(f^\mu)) \nabla H_2(f^\mu) \cdot v \\
- 2 \int_D (K_1 - K_2(f^\mu)) \nabla K_2(f^\mu) \cdot v \\
= 2 \int_w \mu(w) \cdot v(w) - 2 \int_z \tilde{H}(z) \cdot \int_w G(z, w)v(w) - 2 \int_z \tilde{K}(z) \cdot \int_w G(z, w)v(w) \\
= 2 \int_w \{\mu(w) - \int_z [(\tilde{H} + \tilde{K}) \cdot G, \det(\tilde{H} + \tilde{K}, G)] \} \cdot v
\]

Variation obtained from BHF

Iterative Scheme:

\[
\mu^{n+1} - \mu^n = -2(\mu^n - \int_z [(\tilde{H}^n + \tilde{K}^n) \cdot G^n, \det(\tilde{H}^n + \tilde{K}^n, G^n)] )dt
\]

where \( \int_w \cdot := \int_D \cdot dw \) and \( \int_z \cdot := \int_D \cdot dz \) is defined as the integral over the variable \( w \) and \( z \) respectively; \( \tilde{H} := (H_1 - H_2(f^\mu))\nabla H_2(f^\mu); \tilde{K} := (K_1 - K_2(f^\mu))\nabla K_2(f^\mu); \det(a, b) \) is the determinant of the 2 by 2 matrix or equivalently, the norm of the cross product of \( a \) and \( b \).
Quasi-conformal for Registration

Example: Hippocampal registration with geometric matching
Main Goal: Detect abnormal deformation on biological organs
Applications: detecting brain tumor, tracing deformation (medicine evaluation)
Difficulties:
1. Biological organs are geometrically complicated (Example: Brain);
2. Examining abnormalities by the human eye is inefficient & inaccurate.

Goal:
Develop automatic methods to detect & track abnormalities over time.

Tool:
BC: Detecting abnormalities as non-conformal deformation =
1. serious abnormal change;
2. invariant to normal growth [local geometry preserving]
Quasiconformal for Shape Analysis

- Basic idea: *(Lui & Wong et al. 2009)*
  - Compute quasiconformal map (registration) between original and deformed surfaces;
  - Compute its Beltrami coefficients:

\[ |\mu_{\tilde{F}}| = |\mu_{\phi_d^{-1} \circ F \circ \phi_o}| = |\mu_F| \]
Quasiconformal for Shape Analysis

\[ |f^* (ds^2_E) - \text{Identity}| \] (Isometric index)

Isometric indicator: Normal region: [0.75, 1.03]
Abnormal region: [0.08, 1.23]

Beltrami coefficient: Normal region: [0.01, 0.028]
Abnormal region: [0.102, 0.143]
BC effectively shows the region of gyrification!

Brain (Patient)

Brain after gyrification (abnormal deformation)

(Zoom-in)

BC on conformal domain

White = high BC = abnormal!
Quasiconformal for Shape Analysis

Tracking the degree of gyrification over time using Beltrami Index!

Quantitative measurement of gyri thickening!
Quasiconformal for Shape Analysis

BC + Curvatures = complete shape index!

$$E(\mu) = \int_D \alpha |\mu|^2 + \beta (H_1 - H_2(f^\mu))^2 + \gamma (K_1 - K_2(f^\mu))^2$$

Temporal shape changes of healthy and AD HPs:

Normal 1  Normal 2  Alzheimer 1  Alzheimer 2
Shape energy for deformation pattern (from t=0 to t=12 Months)

Statistical significance p-map

(100 normal & 100 AD)
Quasiconformal for **Shape Analysis**

$\text{BC} + \text{Curvatures} = \text{complete shape index!}$

Left = healthy; 
Right = Unhealthy

Beltrami coefficient is not a good shape index for hippocampal shape analysis

Beltrami coefficient + Curvatures is a better shape index

$$E_{\text{shape}}(\mu) = \alpha \int_{D} |\mu|^2 + \beta \int_{D} (H_1 - H_2(f^\mu))^2 + \gamma \int_{D} (K_1 - K_2(f^\mu))^2$$
**Conclusion**

- Computational Conformal/Quasi-conformal Geometry can be used for medical imaging
- More applications on Quasi-conformal geometry in Medical Imaging and computer graphics will be presented by Ronald Lui.

**THANK YOU!**