Expansion Methods for Medical Imaging

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Expansion Methods

- Add structural information or supply missing information to determine specific features with satisfactory resolution and stability.

- Expansion methods used for:
  - scale separations: small targets (potential tumors at early stage), small wave attenuation, small sound speed fluctuations.
  - reconstructions of internal energies in multi-physics imaging from small internal changes in the medium.

- Expansion methods $\Rightarrow$ inversion and optimal design methodologies: analytically investigate the robustness with respect to incomplete data, measurement and medium noises.
Expansion Methods

Small size targets:

- Anomaly detection:
  - boundary and scattering measurements
- Distribution of physical parameters:
  - boundary measurements from internal perturbations of the medium
  - internal measurements
Expansion Methods

Expansion types:

- **Boundary Measurements**: outer expansions in terms of the characteristic size of the anomaly
  - anomaly detection

- **Internal Measurements**: inner expansions
  - distribution of physical parameters
Plan

- Conductivity problem: outer expansions, inner expansions, super-resolved imaging.
- Frequency-domain imaging: compare different imaging functionals.
- Time-domain imaging: time reversal methods, corrections of the effects of attenuation in the medium and random fluctuations in wave speed.
- Concluding remarks.
Conductivity Problem

Notation: $\Omega \in \mathbb{R}^d(d \geq 2)$: smooth bounded domain.

$N(x, z)$: Neumann function for $-\Delta$ in $\Omega$ corresponding to a Dirac mass at $z \in \Omega$:

\[
\begin{cases}
-\Delta_x N(x, z) = \delta_z \quad \text{in } \Omega, \\
\frac{\partial N}{\partial \nu_x} \bigg|_{\partial \Omega} = -\frac{1}{|\partial \Omega|}, \quad \int_{\partial \Omega} N(x, z) \, d\sigma(x) = 0.
\end{cases}
\]

$B$: smooth bounded domain. $\hat{v}$: corrector the solution to

\[
\begin{cases}
\Delta \hat{v} = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B}, \\
\Delta \hat{v} = 0 \quad \text{in } B, \\
\hat{v}|_- - \hat{v}|_+ = 0 \quad \text{on } \partial B, \\
k \frac{\partial \hat{v}}{\partial \nu}|_- - \frac{\partial \hat{v}}{\partial \nu}|_+ = 0 \quad \text{on } \partial B, \\
\hat{v}(\xi) - \xi \to 0 \quad \text{as } |\xi| \to +\infty.
\end{cases}
\]
Conductivity Problem

\[ D = \delta B + z: \text{anomaly} \subset \Omega; \delta: \text{characteristic size of the anomaly}; \text{conductivity} \ 0 < k \neq 1 < +\infty. \]

The voltage potential \( u \):

\[
\begin{align*}
\nabla \cdot \left( \chi(\Omega \setminus \overline{D}) + k\chi(D) \right) \nabla u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega} &= g \quad \left( g \in L^2(\partial\Omega), \int_{\partial\Omega} g \, d\sigma = 0 \right), \\
\int_{\partial\Omega} u \, d\sigma &= 0.
\end{align*}
\]

\( U \): the background solution.
Dipole-type approximation of the conductivity anomaly.

**Theorem:** [Friedman-Vogelius] Outer expansion for $d = 2, 3$:

$$(u - U)(x) \approx -\delta^d \nabla U(z) M(k, B) \nabla z N(x, z) \quad \text{on } \partial \Omega.$$ 

$M(k, B) := (k - 1) \int_B \nabla \hat{v}(\xi) \, d\xi$: the first-order polarization tensor (PT)

The location $z$ and the matrix $M(k, B)$: reconstructed.

$M(k, B)$: - characterizes all the information about the anomaly that can be learned from boundary measurements.
- mixture of $k$ and low-frequency geometric information.

Canonical correspondence between ellipses (ellipsoids) and PTs.

(with Kang, Lecture Notes Math., Springer 2004.)
Polarization Tensor

- Properties of the polarization tensor:
  
  (i) $M$ is symmetric.

  (ii) If $k > 1$, then $M$ is positive definite, and it is negative definite if $0 < k < 1$.

  (iii) Hashin-Shtrikman bounds:

    \[
    \begin{aligned}
    \frac{1}{k - 1} \text{trace}(M) &\leq (d - 1 + \frac{1}{k})|B|, \\
    (k - 1) \text{trace}(M^{-1}) &\leq \frac{d - 1 + k}{|B|}.
    \end{aligned}
    \]

- Optimal size estimates; Thickness estimates; Pólya–Szegö conjecture.

Figure 1: When the two disks have the same radius and the conductivity of the one on the right-hand side is increasing, the equivalent ellipse is moving toward the right anomaly.
Figure 2: When the conductivities of the two disks is the same and the radius of the disk on the right-hand side is increasing, the equivalent ellipse is moving toward the right anomaly.
Theorem: Higher order asymptotic expansion for $x \in \partial \Omega$
(multipolar expansion)

\[ u(x) = U(x) + \sum_{\alpha} \sum_{\beta} \frac{(-1)^{|\beta|\delta^{|\alpha|+|\beta|}}}{\alpha!\beta!} \partial^\alpha U(z) M_{\alpha\beta}(k, B) \partial^\beta N(x, z) + O(\delta^{2d}). \]

- $M_{\alpha\beta}(k, B)$: generalized polarization tensors (GPTs).

- A complete asymptotic expansion is possible. Terms higher than $\delta^{2d-1}$ involves not only GPTs but also the boundary-inclusion interaction.

- $M_{\alpha\beta}(k, B)$ can be obtained from boundary measurements.
Properties of the GPTs

- **Symmetry:** If $\{a_\alpha\}$ and $\{b_\beta\}$ are such that $\sum a_\alpha x^\alpha$ and $\sum b_\beta x^\beta$ are harmonic polynomials, then

$$\sum a_\alpha b_\beta M_{\alpha\beta} = \sum a_\alpha b_\beta M_{\beta\alpha}$$

- **Positivity:** If $k > 1$, then

$$\sum a_\alpha a_\beta M_{\alpha\beta} > 0.$$ 

- **Unique determination of $D$ by GPTs:** If

$$\sum a_\alpha b_\beta M_{\alpha\beta}(k_1, B_1) = \sum a_\alpha b_\beta M_{\alpha\beta}(k_2, B_2) \quad \forall a_\alpha, b_\beta,$$

then $k_1 = k_2$ and $B_1 = B_2$. 
The harmonic moments of $B$ can be estimated from the GPTs.

Wiener-type bounds: Let $k > 1$ and $f = \sum_{\alpha} a_{\alpha} x^{\alpha}$ be a harmonic polynomial. Then

$$(1 - \frac{1}{k}) \int_{B} |\nabla f|^{2} \leq \sum a_{\alpha} a_{\beta} M_{\alpha\beta}(k, B) \leq (k - 1) \int_{B} |\nabla f|^{2}.$$  

(with H. Kang, Polarization and Moment Tensors, Springer 2007).
High-Order Polarization Tensors


Make use of \[ \sum_{|\alpha|+|\beta|\leq K} a_\alpha b_\beta M_{\alpha\beta} \] for a fixed \( K \geq 2 \) to image finer details of the shape of the inclusion and separate material parameter/geometry.

Hopping algorithm in \( |\alpha| + |\beta| \) for finding fine shape details from GPTs: if \( B \) is the target domain, minimize over \( B' \)

\[
\sum_{|\alpha|+|\beta|\leq K} w_{|\alpha|+|\beta|} \left| \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, B') - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, B) \right|^2
\]

\( w_{|\alpha|+|\beta|} \) are binary weights: \( w_{|\alpha|+|\beta|} = 1 \) (on) or 0 (off).

A good choice for the initial guess: the equivalent ellipse.
Figure 3: Reconstruction of clusters of inclusions. The upper images: the equivalent ellipses, and the lower ones: results after $K = 6$ iterations.
High-Order Polarization Tensors

Figure 4: Reconstructions from noisy data using a level set formulation (S. Yu).
Theorem: The following inner asymptotic formula holds:

\[ u(x) \approx U(z) + \delta \hat{v} \left( \frac{x - z}{\delta} \right) \cdot \nabla U(z) \quad \text{for } x \text{ near } z. \]

- Boundary independent reconstruction: no need of an exact knowledge of the boundary of the domain \( \Omega \)
- Local reconstruction
- Separate conductivity/geometry
- Interface approximation: high frequency information
- Trade off between resolution and stability
Internal Measurements

Figure 5: Reconstruction from internal measurements.
Minimization of the discrepancy functional: trade off between resolution and stability:
Reconstruction algorithms and shape representations in wave imaging: dependence on the wavelength $2\pi/\omega$, the Signal-to-noise ration (SNR), and the distance to the target.

Size of the target compared to the wavelength: small.

Far-field measurements.

SNR of an imaging functional $\mathcal{I}$:

$$\text{SNR}(\mathcal{I}) = \frac{\mathbb{E}[\mathcal{I}(z, \omega)]}{\text{Var}(\mathcal{I}(z, \omega))^{1/2}},$$

$z$ the location of the inclusion.
Frequency-Domain Imaging

- $\omega$: frequency; $\mu_0, \epsilon_0$: magnetic permeability, electrical permittivity of the background.
- $\mu, \epsilon$: those of the target $D$.
- $\Gamma_\omega(x)$ be the outgoing Green function for $\Delta + \omega^2$ in $\mathbb{R}^d$.
- Put $k_0 = \omega \sqrt{\epsilon_0 \mu_0}$ and $k = \omega \sqrt{\epsilon \mu}$.
- Incident field: $U(x) = \Gamma_{k_0}(x - y)$.
- Total field: the solution to

$$\nabla \cdot \left( \frac{1}{\mu_0} \chi(\mathbb{R}^d \setminus D) + \frac{1}{\mu} \chi(D) \right) \nabla u(\cdot, y) + \omega^2 \left( \epsilon_0 \chi(\mathbb{R}^d \setminus D) + \epsilon \chi(D) \right) u(\cdot, y) = \frac{1}{\mu_0} \delta_y$$

with the radiation condition imposed on $u$. 
Transmitter and receiver arrays: \( \{y_1, \ldots, y_N\} \).

The multi-static response (MSR) matrix:

\[
(u(y_i, y_j) - \Gamma^k(y_i, y_j))_{i,j=1,\ldots,N}.
\]

Reciprocity property \( \rightarrow \) MSR complex symmetric in the absence of noise (but not Hermitian).

The imaging problem: to reconstruct the target \( D \) and \( \mu, \epsilon \) from the MSR matrix at a single or multiple frequencies.
Theorem: Multipolar asymptotic expansion: \( D = \delta B + z, \) \( x: \) measurement point; \( y: \) source point.

\[
(u - U)(x) = \delta^{d-2} \sum_{p=0}^{\infty} \delta^p \sum_{|\alpha|+|\beta|=p} \frac{1}{\alpha!\beta!} W_{\alpha\beta} \partial_\alpha \Gamma_{k0}^k(z, y) \partial_\beta \Gamma_{k0}^k(x, z)
\]

- \( W_{\alpha\beta} \) depends on \( k, k_0, \) and \( \delta: \) frequency-dependent polarization tensor (FDPT).
- \( \lim_{\delta \to 0} W_{\alpha\beta} \) for \( |\alpha| + |\beta| \leq 1 \) → leading-order term (Vogelius-Volkov)

\[
k_0^2 \left( \frac{\epsilon}{\epsilon_0} - 1 \right) |B| \Gamma_{k0}^k(z, y) \Gamma_{k0}^k(x, z) + M \nabla_z \Gamma_{k0}^k(z, y) \cdot \nabla_z \Gamma_{k0}^k(x, z).
\]

- \( M: \) polarization tensor.
Frequency-Domain Imaging

- **Proposition—Structure of MSR matrix:**
  \[
  A_{ij} := u(y_i, y_j) - \Gamma^{k_0}(y_i, y_j) \approx G(y_i, z)\mathcal{W}G(y_j, z)^T,
  \]

- **Row vector** $G(y_i, z)$:
  \[
  G(y_i, z) = \left(\partial^\alpha_z \Gamma^{k_0}(y_i, z)\right)_{|\alpha| \leq n}.
  \]

  (size $(n + 1)(n + 2)/2$ in $d = 2$; $(n + 1)(n + 2)(n + 3)/6$ in $d = 3$)

- **Matrix** $\mathcal{W}$:
  \[
  \mathcal{W} = \left(\frac{\delta^{d-2+|\alpha|+|\beta|}}{\alpha!\beta!}W_{\alpha\beta}\right)_{|\alpha|,|\beta| \leq n}.
  \]
SVD Structure

SVD of the MSR matrix:

Figure 6: Singular value decomposition of the response matrix $A$ corresponding to two well-separated inclusions of general shape for $N = 20$, using a standard log scale. Six singular values emerge from the 14 others in the noise subspace.
Extended targets: direct imaging not possible. Optimal control approach is efficient.

Figure 7: Singular values of the MSR matrix with 20 transmitters and 20 receivers. The number of significant singular values is about 10 out of 20. Left: unit disk centered at the origin; Right: general shaped target.
Frequency-Domain Imaging

Localization procedure:

- To locate the target:
  - Backpropagation (also called Reverse-Time Migration): $P_m$
  - orthogonal projection onto the $m$ first eigenvectors of MSR matrix,

  $$I_{BP}(z^s) := \sum_{|\alpha| \leq 1} \left\| (I - P_m) \left( \partial^\alpha \Gamma_0 \left( \cdot, z^s \right) \right) \right\|$$

  $$I_{BP}(z^s) := \frac{\sum_{|\alpha| \leq 1} \left\| (I - P_m) \left( \partial^\alpha \Gamma_0 \left( \cdot, z^s \right) \right) \right\|}{\sum_{|\alpha| \leq 1} \left\| P_m \left( \partial^\alpha \Gamma_0 \left( \cdot, z^s \right) \right) \right\|}.$$

- Kirchhoff migration $I_{KM}$, MUSIC $I_{MU}$ or topological derivative based imaging functional $I_{TD}$. 
Imaging Functionals

- $\mathcal{I}_{KM}, \mathcal{I}_{MU}$: subspace weighted projection algorithms.

- Nonlinear relation between $\mathcal{I}_{KM}$ and $\mathcal{I}_{MU}$:
  $\mathcal{I}_{KM} \sim 1/(1 - \mathcal{I}^2_{MU})$.

- $\mathcal{I}_{TD}$: sensitivity of a misfit function relative to the insertion of an inclusion at $z^S$. Maximum point at which the insertion of an inclusion centered at that point maximally decreases the misfit function.

- $\mathcal{I}_{TD}$: backpropagation of the error term in a coherent way.

- $\mathcal{I}_{BP}, \mathcal{I}_{KM}, \mathcal{I}_{MU}, \mathcal{I}_{TD}$: large peaks only at the locations of the inclusions.

- Formulas for the SNR: $\text{SNR}(\mathcal{I}_{KM}) = \text{volume} \times \omega \times \text{contrast} / \text{standard deviation of the noise}$. 
Stability: Backpropagation is the most robust with respect to measurement noise.

Resolution: \( n = 1 \rightarrow \) classical resolution limit.

Resolution: \( n > 1 \rightarrow \) super-resolved imaging if one modifies appropriately the imaging functional (in the regime \( \delta \ll \text{dist} \ll \lambda = 2\pi/k_0 \)).

Modified imaging functional:

\[
\mathcal{I}_{BP}^{(n)}(z^s) := \frac{\sum_{|\alpha|=n} \| (I - P_m)(\partial^\alpha \Gamma^{k_0}(\cdot, \cdot, z^s)) \|}{\sum_{|\alpha|=n} \| P_m(\partial^\alpha \Gamma^{k_0}(\cdot, z^s)) \|}.
\]
Figure 8: Real and imagery part of the BP imaging functional. Here ‘*’ and ‘+’ respectively show the transceiver and the receiver positions.
MUSIC-Type Imaging

- MUSIC Imaging functional:

Figure 9: MUSIC-type reconstruction from the singular value decomposition of $A$ represented in Figure 6.
Imaging Functionals

- Multi-frequency data: summing over frequencies improve imaging via higher effective SNR if independent identically distributed noises.

- Correlation between measurements: single-frequency imaging or Coherent interferometry strategy (CINT) [Borcea-Garnier-Papanicolaou-Tsogka].
Figure 10: Standard deviations of localization error with respect to electronic noise level for $I_{\text{MU}}, I_{\text{BP}}, I_{\text{KM}}, \text{and } I_{\text{TD}}$. 
Figure 11: Realization of a medium noise.
Figure 12: Standard deviations of localization error with respect to clutter noise for $\mathcal{I}_{MU}, \mathcal{I}_{BP}, \mathcal{I}_{KM},$ and $\mathcal{I}_{TD}$. 
Time-Domain Imaging

(with P. Garapon, L. Guadarrama Bustos, and H. Kang, JDE 10)

- $U_y(x, t)$ retarded Green’s function generated at $y \in \Omega$ and $t = 0$ without the anomaly.

- The wave in the presence of the anomaly:

$$\begin{cases} \partial_t^2 u - \nabla \cdot \left( \chi(\mathbb{R}^3 \setminus D) + k\chi(D) \right) \nabla u = \delta_{x=y}\delta_{t=0}, \mathbb{R}^3 \times ]0, +\infty[, \\
  u(x, t) = 0 \quad \text{for} \quad x \in \mathbb{R}^3 \quad \text{and} \quad t \ll 0. \end{cases}$$

- Truncate the high-frequency component of the signal up to

$$\rho = O(\delta^{-\alpha}), \alpha < 1,$$

$$P_{\rho}[u](x, t) = \int_{|\omega| \leq \rho} e^{-i\omega t}\hat{u}(x, \omega)d\omega.$$
\( T = |y - z| \) travel time between the source and the anomaly.

After truncation of the high frequency component, the perturbation due to the anomaly is (approximately) a wave emitted from the point \( z \) at \( t = T \).

Truncation parameter \( \rho \) up to \( O(\delta^{-\alpha}) \), \( \alpha < 1 \).

Far field expansion of \( P_\rho[u - U_y](x, t) \):

\[
= -\delta^3 \int_{\mathbb{R}} \nabla P_\rho[U_z](x, t - \tau) \cdot M(k, B) \nabla P_\rho[U_y](z, \tau) \, d\tau \\
+ O(\delta^{4(1 - \frac{3}{4}\alpha)}) .
\]

The anomaly behaves then like a dipolar source.
Time-Reversal Imaging

- To detect the anomaly from far-field measurements one can use a time-reversal technique.

- One measures the perturbation on a closed surface surrounding the anomaly, truncates its high-frequency component, and retransmits it through the background medium in a time-reversed chronology.

- The perturbation will travel back to the location of the anomaly.
The reversed wave (after high-frequency truncation)

\[ w_{tr}(x, t) = \int_{\mathbb{R}} ds \int_{\partial \Omega} \left[ U_x(x', t - s) \frac{\partial P_{\rho}[u - U_y]}{\partial \nu}(x', t_0 - s) \right. \]
\[ - \left. \frac{\partial U_x}{\partial \nu}(x', t - s) P_{\rho}[u - U_y](x', t_0 - s) \right] d\sigma(x'). \]

(p := \( M(k, B) \nabla P_{\rho}[U_y](z, T) \))

\[ w_{tr}(x, t) \approx -\delta^3 p \cdot \nabla_z \left[ P_{\rho}[U_z](x, t_0 - T - t) - P_{\rho}[U_z](x, t - t_0 + T) \right]. \]

The reversed wave is the sum of incoming and outgoing spherical waves.
Time-Reversal Imaging

- Frequency domain ($\Gamma^\omega$ outgoing Green’s function):

$$
\int_{\partial \Omega} \left[ \Gamma^\omega (x - x') \frac{\partial \Gamma^\omega}{\partial \nu} (z - x') - \overline{\Gamma^\omega} (z - x') \frac{\partial \Gamma^\omega}{\partial \nu} (x - x') \right] d\sigma (x') 
= 2i \Im \Gamma^\omega (z - x) \ \text{ (resolution limit in imaging)}.
$$

- Inhomogeneous media: similar Helmholtz-Kirchhoff identity.

- Resolution limit: size of the focal spot of order half the wavelength.

- Sharper the behavior of $\Im \Gamma^\omega$ at $z$, higher the resolution.

- Super-resolution: dependence of $\Im \Gamma^\omega$ on the heterogeneity of the medium? Local change of the medium around the inclusion.

*(with E. Bonnetier and Y. Capdeboscq, SIAP 09)*
Time Reversal Imaging

■ (with E. Bretin, J. Garnier, A. Wahab, CONM 11)

■ Reconstruct the source term \( f(x) \) from \( g(x, t) := u(x, t) \) on \( \partial \Omega \times [0, T] \),

\[
\begin{aligned}
\partial_t^2 u(x, t) - \Delta u(x, t) &= 0, \quad (x, t) \in \mathbb{R}^d \times [0, T] \\
\partial_t u(x, 0) &= f(x), \quad \partial_t u(x, 0) = 0.
\end{aligned}
\]

■ \( G \) Dirichlet Green function:

\[
\begin{aligned}
\frac{\partial^2 G}{\partial t^2}(x, y, \tau, t) - \Delta_y G(x, y, \tau, t) &= \delta_x \delta_\tau, \quad (y, t) \in \Omega \times \mathbb{R}, \\
G(x, y, \tau, t) &= 0, \quad \partial_t G(x, y, \tau, t) = 0, \quad t \ll \tau,
\end{aligned}
\]

\[
\begin{aligned}
G(x, y, \tau, t) &= 0, \quad (y, t) \in \partial \Omega \times \mathbb{R}.
\end{aligned}
\]
Time Reversal Imaging

- Exact time reversal:

\[ f(x) = \int_0^T \int_{\partial \Omega} \frac{\partial G(x, y, T, t)}{\partial \nu_y} g(y, t - T) d\sigma(y) \quad \forall x \in \Omega. \]

- Modified time reversal: Helmholtz-Kirchhoff identity \( \Rightarrow \) use of the free-space Green function \( U_x \):

\[ f(x) \approx \int_0^T \int_{\partial \Omega} \frac{\partial U_x(y, s, t)}{\partial t} g(y, T - s) d\sigma(y) ds. \]
Attenuation

- Correct the attenuation effect.
- Thermo-viscous model:

\[
\begin{align*}
\partial_t^2 u_a(x, t) - \Delta u_a(x, t) - a\partial_t(\Delta u_a(x, t)) &= 0 \\
u_a(x, 0) &= f(x), \quad \text{and} \quad \partial_t u_a(x, 0) = 0.
\end{align*}
\]

- Attenuation breaks time-reversibility of the wave equation.
- Classical time-reversal produces blurring in reconstructing source terms.
- Regularize the time reversed attenuated wave equation: truncate high frequencies in time or in space.
- Reconstruct the ideal data in the unattenuated medium and then apply classical time-reversal.
\( \tilde{\Gamma}_{a,\omega} \): free space fundamental solution of the Helmholtz equation

\[
\omega^2 \tilde{\Gamma}_{a,\omega}(x, y) + (1 + i\alpha\omega) \Delta \tilde{\Gamma}_{a,\omega}(x, y) = -\delta_x \quad \text{in } \mathbb{R}^d.
\]

Regularized free space fundamental solution:

\[
\tilde{\Gamma}_{a,\rho}(x, y, s, t) = \frac{1}{2\pi} \int_{|\omega| \leq \rho} \tilde{\Gamma}_{a,\omega}(x, y)e^{-i\omega(t-s)} d\omega.
\]

Regularized time reversal:

\[
\mathcal{I}_{2,a,\rho}(x) = \int_{\partial \Omega} \int_0^T \frac{\partial}{\partial t} \tilde{\Gamma}_{a,\rho}(x, y, s, T)g_\alpha(y, T - s)d\sigma(y)ds.
\]
Attenuation

- Relationship between $u_a$ and $u$:

\[
u_a(x, t) = \mathcal{L}_a[u(x, .)](t); \]

\[
\mathcal{L}_a[\varphi](t) = \frac{1}{2\pi} \int \frac{\kappa(\omega)}{\omega} \left\{ \int \varphi(s)e^{i\kappa(\omega)s}ds \right\} e^{-i\omega t}d\omega.
\]

- $\kappa(\omega) = \frac{\omega}{\sqrt{1 - i\alpha \omega}}$.

- Obtain the ideal data $g$ from $g_a$: $\mathcal{L}$ is not well conditioned.
Attenuation

- Regularized inverse of $\mathcal{L}$ via a singular value decomposition (SVD) [La Rivière, Zhang, and Anastasio];

- In physical situations, the coefficient of attenuation $a$ is very small $\Rightarrow$ asymptotic behavior of $\mathcal{L}$ as the attenuation coefficient $a$ tends to zero.

- Stationary Phase Theorem $\Rightarrow$ Asymptotics as $a \to 0$:

$$\mathcal{L}_a[\phi](t) = \phi(t) + \frac{a}{2} (t\phi')'(t) + o(a).$$
Attenuation

- High-order approximation:

\[ \mathcal{L}_a[\phi](t) = \sum_{m=0}^{k} \frac{a^m}{m! 2^m} (t^m \phi')^{(2m-1)}(t) + o(a^k). \]

- An approximation of order \( k \) of the inverse of operator \( \mathcal{L}_a \):

\[ \mathcal{L}_a^{-1} \phi = \sum_{m=0}^{k} a^m \phi_{k,m}(t), \]

- \( \phi_{k,m} \) recursively defined by

\[ \phi_{k,0} = \phi, \quad \phi_{k,m} = - \sum_{l=1}^{m} D_l[\phi_{k,m-l}], \]

- \( D_m \phi(t) = \frac{1}{m! 2^m} (t^m \phi')^{(2m-1)}(t). \)
Step 1: Preprocess the measured data $g_a$ using the filter $L_{a,k}^{-1}$;

Step 2: Use the classical time-reversal functional for the reconstruction of the source $f$. 
Figure 13: Reconstruction by preprocessing the data with the filter $\mathcal{L}_{a,k}^{-1}$; $a = 0.001$; Left: $k = 1$; Right: $k = 4$. 
Attenuation

- Regularized time reversal:

\[ I_{2,a,\rho}(x) = \int_{\partial \Omega} \int_0^T \frac{\partial}{\partial t} \tilde{\Gamma}_{a,\rho}(x, y, s, T) g_a(y, T - s) d\sigma(y) ds. \]

- Regularized time reversal \( I_{2,a,\rho} \) first order correction of the attenuation effect:

\[ I_{2,a,\rho}(x) = \delta_{\rho,x} * f + o(a), \]

- \( \delta_{\rho,x} \): approximation of the Dirac delta distribution.
Introduce the operator:

\[
\tilde{\mathcal{L}}_{a,\rho}[\phi](t) = \frac{1}{2\pi} \int_0^\infty \phi(s) \left\{ \int_{|\omega|\leq\rho} \frac{\tilde{\kappa}(\omega)}{\omega} e^{i\tilde{\kappa}(\omega)s-i\omega t} d\omega \right\} ds,
\]

where \( \tilde{\kappa}(\omega) = \frac{\omega}{\sqrt{1 + i\alpha\omega}} \).

Approximate inverse of \( \mathcal{L}_a \):

\[
\tilde{\mathcal{L}}_{a,\rho}^* \mathcal{L}_a[\phi] = P_{\rho}[\phi] + o(a),
\]

\( P_{\rho} \): truncates high-frequencies.
Cluttered Sound Speed

■ (with E. Bretin, J. Garnier, V. Jugnon, SINUM 11)

■ Wave equation:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(x, t) - c(x)^2 \Delta u(x, t) &= 0, \\
u(x, 0) &= f(x), \\
\frac{\partial u}{\partial t}(x, 0) &= 0.
\end{align*}
\]

■ Unknown cluttered sound speed:

\[
\frac{1}{c(x)^2} = 1 + \sigma_c \mu \left( \frac{x}{x_c} \right),
\]

\(\mu\): stationary random process, \(x_c\): correlation length of the fluctuations of \(c(x)\), \(\sigma_c\): their standard deviation.
Cluttered Sound Speed

- Phases of the measured waves $\hat{u}(\omega, y)$ are shifted with respect to the deterministic, unperturbed phase because of the unknown clutter.
- Numerically back-propagated in the homogeneous medium with speed of propagation equal to one: the phase terms do not compensate each other $\Rightarrow$ blurring and loss of resolution.
- Correct the effect of a clutter: pre-process the data and back-propagate the space and frequency correlations between them.
- CINT (Coherent Interferometry)-Radon algorithm/Kirchhoff-Radon migration.
- Original CINT [Borcea, Garnier, Papanicolaou, Tsogka].
Cluttered Sound Speed

- Two dimensions + \( \partial \Omega \): unit circle.

- Exact inversion formula [L. Nguyen]:

\[
 f = \frac{1}{(2\pi)^3} \int_{\partial \Omega} \int_{\mathbb{R}} B\mathcal{W}[g](y, \omega) e^{-i\omega|x-y|} d\omega d\sigma(y), \quad x \in \Omega,
\]

\[
 \mathcal{W}[g](y, r) := 4r \int_0^r \frac{u(y, t)}{\sqrt{r^2 - t^2}} dt, \quad y \in \partial \Omega, \quad r \in \mathbb{R}^+,
\]

- \( B \): filter

\[
 B[h](y, t) = \int_0^2 \frac{d^2 h(y, r)}{dr^2} (y, r) \ln(|r^2 - t^2|) dr, \quad y \in \partial \Omega.
\]
Cluttered Sound Speed

- Pre-processed data: \( q = \frac{1}{4\pi^2} \mathcal{B}\mathcal{W}[g] \).

- Kirchhoff-Radon migration imaging function:

\[
\mathcal{I}_{\text{KRM}}(x) = \frac{1}{2\pi} \int_{\partial\Omega} d\sigma(y) \int_{\mathbb{R}} d\omega \hat{q}(x, \omega) e^{-i\omega|x-y|}.
\]

- CINT-Radon imaging function:

\[
\mathcal{I}_{\text{CIR}}(x) = \frac{1}{(2\pi)^2} \int_{\partial\Omega} \int_{\partial\Omega} d\sigma(y_1) d\sigma(y_2) \\
\times \int_{\mathbb{R} \times \mathbb{R}} d\omega_1 d\omega_2 e^{-\frac{(\omega_2-\omega_1)^2}{2\Omega^2} - \frac{|y_1-y_2|^2}{2x^2}} \\
\times \hat{q}(y_1, \omega_1) e^{-i\omega_1|x-y_1|} \hat{q}(y_2, \omega_2) e^{i\omega_2|x-y_2|}.
\]
CINT-Radon: compute the local correlation of the pre-processed data in a time interval scaled by $\frac{1}{\Omega_d}$ and superpose the back-propagated local correlations over pairs of receivers that are not further apart from $X_d$.

CINT-Radon: keep the pairs $(y_1, \omega_1)$ and $(y_2, \omega_2)$ for which the pre-processed data $\hat{q}(y_1, \omega_1)$ and $\hat{q}(y_2, \omega_2)$ are coherent and remove the pairs that do not bring information.

If $\Omega_d \to \infty$ and $X_d \to \infty$, then $I_{CIR}$ is the square of the Kirchhoff-Radon migration function:

$$I_{CIR}(x) = |I_{KRM}(x)|^2.$$
Cluttered Sound Speed

- Kirchhoff-Radon: strong damping and very small SNR.
- CINT-Radon: high SNR and choice of the cut-off parameters \( \Omega_d \approx \) frequency coherence radius and \( X_d \approx \) spatial coherence radius achieves a good trade-off between resolution and stability. Smaller values increase SNR but reduces the resolution.
- Optimize over \( \Omega_d \) and \( X_d \) the quality of the image [as in adaptive CINT, Borcea-Papanicolaou-Tsogka].
Cluttered Sound Speed

Figure 14: Random velocity.
Figure 15: Source reconstructions using $I_KRM$ (left) and $I_{CIR}$, with $X_d = 2$ and $\Omega_d = 200$ (right).
Cluttered Sound Speed

Figure 16: Left: point sources; right: random velocity with high frequencies.

Figure 17: Left: Kirchhoff reconstruction; right: CINT reconstruction.
Multi-physics Imaging

- One single imaging system based on the combined use of two kinds of waves.
- One wave will give its contrast and the second its spatial resolution.
Multi-physics Imaging

(with E. Bonnetier, Y. Capdeboscq, M. Tanter, and M. Fink, SIAP 08)

$u$ the voltage potential induced by a current $g$ in the absence of acoustic perturbations:

$$\nabla_x \cdot (\gamma(x) \nabla_x u) = 0 \text{ in } \Omega,$$

$$\gamma(x) \frac{\partial u}{\partial \nu} = g \text{ on } \partial \Omega,$$

Suppose $\gamma$ bounded from below and above and known in a neighborhood of the boundary $\partial \Omega$: $\gamma = \gamma_*$; Set $\Omega' \subset \Omega$ where $\gamma$ is unknown.
Multi-physics Imaging

Use 2 different amplitudes of focalized ultrasonic waves (with the same focal spot $D$) →

$$\gamma_{\delta}(x) = \gamma(x) \left[ 1 + \chi(D)(x) (\nu(x) - 1) \right],$$

with $\nu(x)$: known.

$u_{\delta}$ induced by $g$ in the presence of acoustic perturbations localized in the focal spot $D := z + \delta B$:

$$\begin{cases} 
\nabla_x \cdot (\gamma_{\delta}(x) \nabla_x u_{\delta}(x)) = 0 \text{ in } \Omega, \\
\gamma(x) \frac{\partial u_{\delta}}{\partial \nu} = g \text{ on } \partial \Omega.
\end{cases}$$
Multi-physics Imaging

**Theorem:** Suppose the focal spot $D$ to be a disk and $u \in W^{2,\infty}(D)$. Then,

$$\int_{\partial \Omega} (u_{\delta} - u)g d\sigma = \left| \nabla u(z) \right|^2 \int_D \gamma(x) \frac{(\nu(x) - 1)^2}{\nu(x) + 1} dx$$

$$+ O(|D|^{1+\kappa}),$$

$$O(|D|^{1+\kappa}) \leq C |D|^{1+\kappa} \left\| \nabla u \right\|_{L^{\infty}(D)} \left\| \nabla^2 u \right\|_{L^{\infty}(D)}$$

with $C'$: independent of $D$ and $u$.

$\kappa$: depends only on $\Omega'$, $\nu$, $\sup_{\Omega} \gamma$, $\min_{\Omega} \gamma$. 

INI – p. 65/72
Corollary: Suppose $\gamma \in C^{0,\alpha}(D)$, $0 \leq \alpha \leq 2\kappa \leq 1$. Then

$$E(z) := \left( \int_D \frac{(\nu(x) - 1)^2}{\nu(x) + 1} \, dx \right)^{-1} \int_{\partial\Omega} (u_\delta - u) g \, d\sigma$$

$$= \gamma(z) |\nabla u(z)|^2 + O(|D|^{\alpha/2}).$$

- $E(z)$: known function from the boundary measurements;
- $\gamma(z) |\nabla u(z)|^2$: electrical energy density;
- Focal spot $D$ not a disk: slightly different approximation for the electrical energy density;
- $\alpha = 0 \rightarrow$ remainder $o(1)$. 
Substitute $\gamma$ by $\mathcal{E}/|\nabla u|^2$.

Nonlinear PDE (the 0–Laplacian)

\[
\begin{align*}
\nabla_x \cdot \left( \frac{\mathcal{E}}{|\nabla u|^2} \nabla u \right) &= 0 \quad \text{in } \Omega, \\
\frac{\mathcal{E}}{|\nabla u|^2} \frac{\partial u}{\partial \nu} &= g \quad \text{on } \partial \Omega.
\end{align*}
\]

$g$ such that $u$ has no critical point inside $\Omega'$.

Choose two currents $g_1$ and $g_2$ s.t. $\nabla u_1 \times \nabla u_2 \neq 0$ for all $x \in \Omega$. 
Multi-physics Imaging

(with Y. Capdeboscq, F. de Gournay, A. Rozanova, and F. Triki, preprint)

Model problem: The Helmholtz equation

\[
\begin{cases}
\nabla \cdot (\gamma \nabla u) + k^2 q u = 0 \text{ in } \Omega, \\
\gamma \frac{\partial u}{\partial \nu} = g \text{ on } \partial \Omega.
\end{cases}
\]

Reconstruct both \( \gamma \) and \( q \) for fixed frequency \( k \) (not a resonant frequency).

Focalized ultrasonic waves: local change in the electromagnetic parameters of the medium \( \gamma \) and \( q \), linearly with respect to the amplitude of the ultrasonic signal.

Generalize the 0-Laplacian approach.
Use 4 different amplitudes of ultrasonic waves (with the same focal spot) \( \gamma_{\delta}(x) = \gamma(x) \left[ 1 + \chi(D)(x) (\nu_1(x) - 1) \right] \), and

\[
q_{\delta}(x) = q(x) \left[ 1 + \chi(D)(x) (\nu_2(x) - 1) \right], \text{ with } \nu_i(x): \text{ known.}
\]

**Theorem:** Focal spot \( D \): disk; \( \gamma \) and \( q \in C^{0,\alpha}(D) \), \( u \in W^{2,\infty}(D) \);

\[
\int_{\partial \Omega} (u_{\delta} - u) \bar{g} d\sigma = \gamma(z) |\nabla u(z)|^2 \int_D \frac{(\nu_1(x) - 1)^2}{\nu_1(x) + 1} dx
\]

\[
+ k^2 q(z) |u(z)|^2 \int_D (\nu_2(x) - 1) dx + O(|D|^{1+\alpha/2}).
\]

\( \alpha = 0 \rightarrow \text{remainder } o(|D|). \)
Multi-physics Imaging

- Obtain $\mathcal{E}_\gamma := \gamma |\nabla u|^2$ and $\mathcal{E}_q := q|u|^2$ from boundary measurements.
- Reconstruct $\gamma$ and $q$ from $\mathcal{E}_\gamma$ and $\mathcal{E}_q$.
- Nonlinear PDE (the modified 0–Laplaceian)

\[
\left\{ \begin{array}{l}
\nabla \cdot \left( \frac{\mathcal{E}_\gamma}{|\nabla u|^2} \nabla u \right) + k^2 \frac{\mathcal{E}_q}{|u|^2} u = 0 \quad \text{in } \Omega , \\
\frac{\mathcal{E}_\gamma}{|\nabla u|^2} \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial \Omega .
\end{array} \right.
\]
Reconstruct the electromagnetic parameters knowing the electromagnetic internal energies:

- Linearized versions of the nonlinear (zero-Laplacian) PDE problems.
- Optimal control approach: minimize over the electromagnetic parameters the discrepancy between the computed and reconstructed internal energies.
- Optimal control approach: more efficient approach specially with incomplete internal measurements of the electromagnetic energy densities.
- Resolution of order the size of the focal spot + stability (wrt measurement noise).

Exact inversion formulas are used to obtain a good initial guess.
Conclusion

- Expansion methods: power strategy for increasing stability and resolution.
- Expansion methods: combine a variety of mathematical and computational tools.
- Extension to elasticity (concept of elastic moment tensor and viscous moment tensor, inner expansions for elastography, weighted Helmholtz decomposition for time reversal imaging).


- Expansion methods: story has still long way to go!