D-optimal designs for Two-Variable Binary Logistic Models with Interaction

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Problem
Setting

- Binary response $p = P(\text{success})$
Binary response $p = P(\text{success})$

Two explanatory variables $x_1$ and $x_2$ with interaction
Setting

- Binary response $p = P(\text{success})$

- Two explanatory variables $x_1$ and $x_2$ with interaction

- Locally $D$-optimal designs?
Model

\[ \text{logit}(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \]
Model

- $\text{logit}(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$

- Interaction $\beta_{12}$:
  - $< 0$ antagonistic
  - $= 0$ additive
  - $> 0$ synergistic
Model

- \( \text{logit}(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \)

- Interaction \( \beta_{12} \):
  - \(< 0\) antagonistic
  - \(= 0\) additive
  - \(> 0\) synergistic

- Relative potency \( \rho = \frac{\beta_2}{\beta_1} \)
Earlier Studies
Earlier Studies

Earlier Studies


Earlier Studies


Multiplicity of designs
First Take

- Multiplicity of designs
- Taxonomy?
First Take

- Multiplicity of designs
- Taxonomy?
- Algebra?
Preliminaries
No Interaction Model
No Interaction Model

\[ u = \logit(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \]
No Interaction Model

- $u = \text{logit}(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

- Lines of constant logit
D-optimal Design (1): \(-b \leq x_2 \leq b\)
**D-optimal Design (1):** \(-b \leq x_2 \leq b\)

\((\pm u^*, \pm b)\)
D-optimal Design (1): \(-b \leq x_2 \leq b\)

\((\pm u^*, \pm b)\)
What is $u^*$?
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- $k$-variables:
What is $u^*$?

- $k$-variables: $u^*$ solves

$$(k + 1)u + 2 - ((k + 1)u - 2)e^u = 0$$
What is $u^*$?

- $k$-variables: $u^*$ solves

\[(k + 1)u + 2 - ((k + 1)u - 2)e^u = 0\]

- Proof
What is $u^*$?

• *$k$-variables*: $u^*$ solves

$$(k + 1)u + 2 - ((k + 1)u - 2)e^u = 0$$

• Proof

  • Yang, Zhang and Huang (2011)
What is $u^*$?

- **$k$-variables:** $u^*$ solves

\[(k + 1)u + 2 - ((k + 1)u - 2)e^u = 0\]

- **Proof**

  - Yang, Zhang and Huang (2011)
  - Kabera, Haines and Ndlovu (2011)
What is $u^*$?

- **$k$-variables:** $u^*$ solves
  
  $$(k + 1)u + 2 - ((k + 1)u - 2)e^u = 0$$

- **Proof**
  - Yang, Zhang and Huang (2011)
  - Kabera, Haines and Ndlovu (2011)

- $k = 2$, $u^* = 1.2229$ and $k = 1$, $u^* = 1.5434$
D-optimal Design (2a): $x_1 \geq 0, x_2 \geq 0$
**D-optimal Design (2a):** \( x_1 \geq 0, x_2 \geq 0 \)

\[ \beta_0 < -1.5434 \]
D-optimal Design (2a): \( x_1 \geq 0, \ x_2 \geq 0 \)

\( \beta_0 < -1.5434 \)
**D-optimal Design (2b):** \[ x_1 \geq 0 , x_2 \geq 0 \]
\textbf{D-optimal Design (2b)}: \( x_1 \geq 0, \ x_2 \geq 0 \)

\[ \beta_0 > -1.5434 \]
**D-optimal Design (2b):** \( x_1 \geq 0, x_2 \geq 0 \)

\[ \beta_0 > -1.5434 \]
Results
Approach
Approach

- Model

\[ u = \logit(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \]
Approach

● Model

\[ u = \logit(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \]

● Transform

\[ z_1 = x_1 + \frac{\beta_2}{\beta_{12}} \quad \text{and} \quad z_2 = x_2 + \frac{\beta_1}{\beta_{12}} \]
Approach

● Model

\[ u = \logit(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \]

● Transform

\[ z_1 = x_1 + \frac{\beta_2}{\beta_{12}} \quad \text{and} \quad z_2 = x_2 + \frac{\beta_1}{\beta_{12}} \]

● Then

\[ u = \beta_0^* + \beta_{12} z_1 z_2 \]
Approach

- **Model**

\[ u = \text{logit}(p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \]

- **Transform**

\[ z_1 = x_1 + \frac{\beta_2}{\beta_{12}} \quad \text{and} \quad z_2 = x_2 + \frac{\beta_1}{\beta_{12}} \]

- **Then**

\[ u = \beta^*_0 + \beta_{12} z_1 z_2 \]

where \( \beta^*_0 = \beta_0 - \frac{\beta_1 \beta_2}{\beta_{12}} \)
Assume $\beta_1 > 0$ and $\beta_2 > 0$ so that $\beta_0^* < \beta_0$
Assume $\beta_1 > 0$ and $\beta_2 > 0$ so that $\beta^*_0 < \beta_0$

Interchange

$$(x_1, x_2) \leftrightarrow (z_1, z_2) \leftrightarrow (u, x_2) \leftrightarrow (u, z_2)$$

where $u = \beta_0 + b_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 = \beta^*_0 + \beta_{12} z_1 z_2$
Assume $\beta_1 > 0$ and $\beta_2 > 0$ so that $\beta_0^* < \beta_0$

Interchange

$$(x_1, x_2) \leftrightarrow (z_1, z_2) \leftrightarrow (u, x_2) \leftrightarrow (u, z_2)$$

where $u = \beta_0 + b_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 = \beta_0^* + \beta_{12} z_1 z_2$

Logit lines

$x_1 = 0, x_2 = 0$ lies on $u = \beta_0$

$z_1 = 0, z_2 = 0$ lies on $u = \beta_0^*$
Lines of Constant Logit: $u > \beta_0^*$
Lines of Constant Logit: $u > \beta^*_0$
Lines of Constant Logit: $u < \beta_0^*$
Lines of Constant Logit: $u < \beta^*_0$
D-optimal Design (1): \( z_2 = \pm b \)
D-optimal Design (1): \( z_2 = \pm b \)
Design Details

- Fix $z_2 = +b$ and $z_2 = -b$
Design Details

- Fix \( z_2 = +b \) and \( z_2 = -b \)

- Design \( (\pm u^*, \pm b) \) with \( u^* = 1.5434 \) and \( \beta_0^* > u^* \)
Design Details

- Fix $z_2 = +b$ and $z_2 = -b$

- Design $(\pm u^*, \pm b)$ with $u^* = 1.5434$ and $\beta_0^* > u^*$

- Why?
**Design Details**

- Fix $z_2 = +b$ and $z_2 = -b$

- Design $(\pm u^*, \pm b)$ with $u^* = 1.5434$ and $\beta_0^* > u^*$

**Why?**

- Proof
Design Details

- Fix $z_2 = +b$ and $z_2 = -b$

- Design $(\pm u^*, \pm b)$ with $u^* = 1.5434$ and $\beta_0^* > u^*$

- Why?
  - Proof
  - Obvious!
D-optimal Design (2a): \( x_1 \geq 0, x_2 \geq 0 \)
D-optimal Design (2a): \( x_1 \geq 0, x_2 \geq 0 \)
**D-optimal Design (2a):** \( x_1 \geq 0, x_2 \geq 0 \)

\( \beta_1 > 0 \) and \( \beta_2 > 0 \)
D-optimal Design (2a): $x_1 \geq 0, x_2 \geq 0$

$\beta_1 > 0$ and $\beta_2 > 0$
**Condition:** $\beta_0^* < \beta_0 < -1.5434$
Design Details

- Condition: $\beta^*_0 < \beta_0 < -1.5434$

- Design points $(x_1, x_2)$
Design Details

- Condition: $\beta^*_0 < \beta_0 < -1.5434$

- Design points $(x_1, x_2)$
  - $(0, 0)$
Design Details

- Condition: $\beta_0^* < \beta_0 < -1.5434$

- Design points $(x_1, x_2)$
  - $(0, 0)$
  - $(0, a_1)$ and $(a_2, 0)$ on the same logit line
Design Details

- Condition: $\beta^*_0 < \beta_0 < -1.5434$

- Design points $(x_1, x_2)$
  - $(0, 0)$
  - $(0, a_1)$ and $(a_2, 0)$ on the same logit line
  - $(x_1, \frac{\beta_1}{\beta_2} x_1)$ on the ray with slope $\frac{1}{\rho}$
Design Details

- **Condition:** $\beta^*_0 < \beta_0 < -1.5434$

- **Design points** $(x_1, x_2)$
  - $(0, 0)$
  - $(0, a_1)$ and $(a_2, 0)$ on the same logit line
  - $(x_1, \frac{\beta_1}{\beta_2} x_1)$ on the ray with slope $\frac{1}{\rho}$

- **Why?**
**D-optimal Design (2b):**  \[ x_1 \geq 0, \ x_2 \geq 0 \]
D-optimal Design (2b): \( x_1 \geq 0, x_2 \geq 0 \)

\( \beta_1 > 0 \) and \( \beta_2 > 0 \)
**D-optimal Design (2b):** \( x_1 \geq 0, \ x_2 \geq 0 \)

\( \beta_1 > 0 \) and \( \beta_2 > 0 \)
Design Details
**Condition:** $\beta_0^* < -1.5434 < \beta_0$ or $-1.5434 < \beta_0^* < \beta_0$
Design Details

- Condition: $\beta_0^* < -1.5434 < \beta_0$ or $-1.5434 < \beta_0^* < \beta_0$

- Design points $(x_1, x_2)$
Design Details

- **Condition:** $\beta_0^* < -1.5434 < \beta_0$ or $-1.5434 < \beta_0^* < \beta_0$

- **Design points** $(x_1, x_2)$
  - $(0, a_1)$ and $(a_2, 0)$ on the same logit line
Condition: $\beta_0^* < -1.5434 < \beta_0$ or $-1.5434 < \beta_0^* < \beta_0$

Design points $(x_1, x_2)$

- $(0, a_1)$ and $(a_2, 0)$ on the same logit line
- $(0, b_1)$ and $(b_2, 0)$ on the same logit line
Design Details

● Condition: $\beta^*_0 < -1.5434 < \beta_0$ or $-1.5434 < \beta^*_0 < \beta_0$

● Design points $(x_1, x_2)$
  • $(0, a_1)$ and $(a_2, 0)$ on the same logit line
  • $(0, b_1)$ and $(b_2, 0)$ on the same logit line
  • Two points on the ray with slope $\frac{1}{\rho}$
● Condition: $\beta^*_0 < -1.5434 < \beta_0$ or $-1.5434 < \beta^*_0 < \beta_0$

● Design points $(x_1, x_2)$
  - $(0, a_1)$ and $(a_2, 0)$ on the same logit line
  - $(0, b_1)$ and $(b_2, 0)$ on the same logit line
  - Two points on the ray with slope $\frac{1}{\rho}$

● Proof?
Conclusions
Binary response in two variables
Binary response in two variables

- Algebra and patterns → proofs
Binary response in two variables

- Algebra and patterns → proofs

- Taxonomy for $x_1 \geq 0, x_2 \geq 0$
Binary response in two variables

- Algebra and patterns $\rightarrow$ proofs

- Taxonomy for $x_1 \geq 0, x_2 \geq 0$
  - $\beta_1 > 0$ and $\beta_2 < 0$
Binary response in two variables

- Algebra and patterns $\rightarrow$ proofs

- Taxonomy for $x_1 \geq 0$, $x_2 \geq 0$
  - $\beta_1 > 0$ and $\beta_2 < 0$
  - $\beta_{12} < 0$
Binary response in two variables

- Algebra and patterns $\rightarrow$ proofs

- Taxonomy for $x_1 \geq 0, x_2 \geq 0$
  - $\beta_1 > 0$ and $\beta_2 < 0$
  - $\beta_{12} < 0$

- Further constraints on $x_1$ and $x_2$