Differential operators on quantized flag manifolds

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- G: connected simply-connected simple algebraic group over $\mathbf{C}$ ($\mathfrak{g} = \mathrm{Lie}\, G$),

- Q: root lattice,
- P: weight lattice,
- $P^+ = \{$dominant weights$\}$,

- $U_q(\mathfrak{g})$ =(quantized enveloping algebra of $\mathfrak{g}$ over $\mathbf{F}$)
  $= \langle K_\lambda, E_i, F_i \mid \lambda \in P,\ i \in I \rangle$
- $\mathbf{F} = \mathbf{C}(q^{1/N}), \qquad N = |P/Q|$

ordinary flag manifold

$$\mathcal{B} = B^-\backslash G = \mathrm{Proj}\,(\mathbf{C}[N^-\backslash G]),$$
$$\mathbf{C}[N^-\backslash G] = \{f \in \mathbf{C}[G] \mid f(ng) = f(g)\ (n \in N^-,\ g \in G)\}$$
$$= \bigoplus_{\lambda \in P^+} \mathbf{C}[N^-\backslash G](\lambda) \quad (P\text{-graded } \mathbf{C}\text{-algebra}).$$

quantized flag manifold

We have analogues $\mathbf{C}_q[G]$ and $\mathbf{C}_q[N^-\backslash G]$ of $\mathbf{C}[G]$ and $\mathbf{C}[N^-\backslash G]$ respectively (Hopf algebra over $\mathbf{F}$ and P-graded $\mathbf{F}$-algebra).
We set

$$\mathcal{B}_q = \mathrm{Proj}\,(\mathbf{C}_q[N^-\backslash G]).$$

$\mathcal{B}_q$ is a non-commutative projective scheme with non-commutative homogeneous coordinate algebra $\mathbf{C}_q[N^-\backslash G]$.

quantized flag manifolds

We do not have "a space" $\mathcal{B}_q$,
but we have a category

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_q}) = \{\text{“quasi-coherent } \mathcal{O}_{\mathcal{B}_q}\text{-module”}\}.$$

remark

An ordinary (commutative) scheme X is uniquely determined by the category $\mathrm{Mod}(\mathcal{O}_X)$ of quasi-coherent sheaves.

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_q}) = \frac{\mathrm{Mod}_{gr}(\mathbf{C}_q[N^-\backslash G])}{\mathrm{Tor}(\mathbf{C}_q[N^-\backslash G])},$$

$$\mathrm{Mod}_{gr}(\mathbf{C}_q[N^-\backslash G]) = \{\text{graded left } \mathbf{C}_q[N^-\backslash G]\text{)-module}\},$$

$$\mathrm{Tor}(\mathbf{C}_q[N^-\backslash G]) = \{M \in \mathrm{Mod}_{gr}(\mathbf{C}_q[N^-\backslash G]) \mid \forall m \in M \quad \mathbf{C}_q[N^-\backslash G](\lambda)m = 0 \ (\lambda \gg 0)\}.$$

$\mathcal{D}$-modules on quantized flag manifolds

For each $t \in H(\mathbf{F})$ we can define a category

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_q,t}) = \{\text{"quasi-coherent } \mathcal{D}_{\mathcal{B}_q,t}\text{-module"}\}$$

as follows (H is a maximal torus of G).
Define a subalgebra D_q of $\mathrm{End}_{\mathbf{F}}(\mathbf{C}_q[N^-\backslash G])$ by

$$D_q = \langle\, \ell_\varphi,\ \partial_u,\ \sigma_\lambda \mid \varphi \in \mathbf{C}_q[N^-\backslash G],\ u \in U_q(\mathfrak{g}),\ \lambda \in P\,\rangle_{\mathrm{alg}},$$

$$\ell_\varphi(\psi) = \varphi\psi,$$
$$\partial_u(\psi) = u \cdot \psi,$$
$$\sigma_\lambda(\psi) = q^{(\lambda,\mu)}\psi \quad (\psi \in \mathbf{C}_q[N^-\backslash G])(\mu)).$$

D_q is a P-graded $\mathbf{F}$-algebra by

$$D_q(\lambda) = \{F \in D_q \mid F(\mathbf{C}_q[N^-\backslash G])(\mu)) \subset \mathbf{C}_q[N^-\backslash G])(\mu + \lambda)\}.$$

Then

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_q,t}) = \frac{\mathrm{Mod}_{gr,t}(D_q)}{\mathrm{Mod}_{gr,t}(D_q) \cap \mathrm{Tor}(\mathbf{C}_q[N^-\backslash G])},$$

$$\mathrm{Mod}_{gr,t}(D_q) = \{M\text{: graded } D_q\text{-module} \mid \sigma_\lambda|_{M(\mu)} = q^{(\lambda,\mu)}\chi_\lambda(t)\,\mathrm{id}\}.$$

analogue of the Beilinson-Bernstein correspondence

We have an embedding $P \subset H(\mathbf{F})$ given by

$$\lambda \mapsto t_\lambda, \qquad \chi_\mu(t_\lambda) = q^{(\lambda,\mu)}.$$

Theorem (modified Lunts-Rosenberg conjecture)

For $t \in P^+$ we have an equivalence

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_q,t}) \cong \mathrm{Mod}(U_q(\mathfrak{g})/U_q(\mathfrak{g})\,\mathrm{Ker}\,c_t)$$
$$(c_t : (\text{center of } U_q(\mathfrak{g})) \to \mathbf{F}).$$

of abelian categories.

remark

(center of $U_q(\mathfrak{g})$) $\cong \mathbf{F}[2P]^W$.

Specialization of q at roots of 1

Set

$$\mathbf{A} = \mathbf{C}[q^{1/N}, q^{-1/N}] \subset \mathbf{F} = \mathbf{C}(q^{1/N}).$$

We consider the specialization

$$\mathbf{A} \to \mathbf{C} \qquad (q^{1/N} \mapsto \zeta = \exp(2\pi\sqrt{-1}/\ell)),$$

where ℓ satisfies

(a) ℓ is odd, $\ell > 1$

(b) ℓ is prime to 3 for G_2,

(c) ℓ is prime to $N = |P/Q|$.

By (c) $q \mapsto \zeta^N =$ (primitive ℓ-th root of 1).

quantized enveloping algebras at roots of 1

We have two $\mathbf{A}$-forms of $U_q(\mathfrak{g})$;

$$U_{\mathbf{A}}^{DK}(\mathfrak{g}) = \langle K_\lambda, E_i, F_i \mid \lambda \in P, i \in I \rangle_{\mathbf{A}-\text{algebra}} \subset U_q(\mathfrak{g})$$

(De Concini-Kac form),

$$U_{\mathbf{A}}^{L}(\mathfrak{g}) = \langle K_\lambda, E_i^{(n)}, F_i^{(n)} \mid \lambda \in P, i \in I, n \in \mathbf{N} \rangle_{\mathbf{A}-\text{algebra}} \subset U_q(\mathfrak{g})$$

(Lusztig form).

We set

$$U_\zeta^{DK}(\mathfrak{g}) = \mathbf{C} \otimes_{\mathbf{A}} U_{\mathbf{A}}^{DK}(\mathfrak{g}),$$
$$U_\zeta^{L}(\mathfrak{g}) = \mathbf{C} \otimes_{\mathbf{A}} U_{\mathbf{A}}^{L}(\mathfrak{g}).$$

We have specializations $\mathbf{C}_\zeta[G]$ and $\mathbf{C}_\zeta[N^-\backslash G]$ of ($\mathbf{A}$-forms of) $\mathbf{C}_q[G]$ and $\mathbf{C}_q[N^-\backslash G]$ respectively ($\mathbf{C}_\zeta[G]$ is dual to $U^L_\zeta(\mathfrak{g})$).
We set

$$\mathcal{B}_\zeta = \mathrm{Proj}(\mathbf{C}_\zeta[N^-\backslash G]).$$

$\mathcal{B}_\zeta$ is a non-commutative projective scheme with non-commutative homogeneous coordinate algebra $\mathbf{C}_\zeta[N^-\backslash G]$.
We have a category

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) = \frac{\mathrm{Mod}_{gr}(\mathbf{C}_\zeta[N^-\backslash G])}{\mathrm{Tor}(\mathbf{C}_\zeta[N^-\backslash G])},$$

$$\mathrm{Mod}_{gr}(\mathbf{C}_\zeta[N^-\backslash G]) = \{\text{graded left } \mathbf{C}_\zeta[N^-\backslash G]\text{)-module}\},$$

$$\mathrm{Tor}(\mathbf{C}_\zeta[N^-\backslash G]) = \{M \in \mathrm{Mod}_{gr}(\mathbf{C}_\zeta[N^-\backslash G]) \mid \forall m \in M \quad \mathbf{C}_\zeta[N^-\backslash G](\lambda)m = 0 \ (\lambda \gg 0)\}.$$

We can deal with $\mathcal{B}_\zeta$ in the framework of the ordinary (commutative) geometry as follows.

Frobenius twist

We have a central embedding

$$\mathbf{C}[G] \to \mathbf{C}_\zeta[G],$$

of Hopf algebras (Lusztig),
which induces

$$\mathbf{C}[N^-\backslash G] \to \mathbf{C}_\zeta[N^-\backslash G],$$

and the Frobenius twist

$$Fr : \mathcal{B}_\zeta \to \mathcal{B} = B^-\backslash G.$$

We obtain a non-commutative $\mathcal{O}_\mathcal{B}$-algebra $Fr_*\mathcal{O}_{\mathcal{B}_\zeta}$, for which we have

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \cong \mathrm{Mod}(Fr_*\mathcal{O}_{\mathcal{B}_\zeta}).$$

Similarly to $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_q,t})$ we can define, for $t \in H$, a category $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ as follows.
Set

$$D_{\mathbf{A}} = \langle\, \ell_\varphi,\ \partial_u,\ \sigma_\lambda \mid \varphi \in \mathbf{C}_{\mathbf{A}}[N^-\backslash G],\ u \in U_{\mathbf{A}}^{DK}(\mathfrak{g}),\ \lambda \in P\,\rangle_{\mathbf{A}\text{-alg}} \subset D_q,$$

$$D_\zeta = \mathbf{C} \otimes_{\mathbf{A}} D_{\mathbf{A}}.$$

Then

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) = \frac{\mathrm{Mod}_{gr,t}(D_\zeta)}{\mathrm{Mod}_{gr,t}(D) \cap \mathrm{Tor}(\mathbf{C}_\zeta[N^-\backslash G])},$$

$$\mathrm{Mod}_{gr,t}(D_\zeta) = \{M\text{: graded } D_\zeta\text{-module} \mid \sigma_\lambda|_{M(\mu)} = q^{(\lambda,\mu)}\chi_\lambda(t)\,\mathrm{id}\}.$$

As in the case of $\mathcal{O}$-modules, we have

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) \cong \mathrm{Mod}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})$$

for some non-commutative $\mathcal{O}_{\mathcal{B}}$-algebra $Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$.

A description of $Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$

Note

$$D_\zeta = \bigoplus_{\lambda \in P^+} D_\zeta(\lambda).$$

Set

$$D_\zeta^{(\ell)} = \bigoplus_{\lambda \in P^+} D_\zeta(\ell\lambda), \qquad D_{\zeta,t}^{(\ell)} = D_\zeta^{(\ell)} \otimes_{\mathbf{C}[P]} \mathbf{C}_t.$$

Then $D_{\zeta,t}^{(\ell)}$ is a graded algebra containing $\mathbf{C}[N^- \backslash G]$ as a central graded subalgebra.

$Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$ is the $\mathcal{O}_{\mathcal{B}}$-algebra corresponding to $D_{\zeta,t}^{(\ell)}$.

Aim

Give analogues of results of Bezrukavnikov-Mirkovič-Rumynin about $\mathcal{D}$-modules on flag manifolds in positive characteristics.

Results of Bezrukavnikov-Mirkovic-Rumynin

Notation

k : algebraically closed field of characteristic $p > 0$,

G_k : connected simply-connected simple algebraic group over k

$(p >$ (Coxeter number)$)$,

$\mathcal{B}_k$: flag manifold for G_k

$\mathcal{D}_{\mathcal{B}_k,\lambda}$: the sheaf of twisted differential operators on $\mathcal{B}_k$ $(\lambda \in \mathfrak{h}_k^*)$

Theorem A (BMR)

If $\lambda \in \mathfrak{h}_k^*$ is regular, then

$$D^b(\mathrm{Mod}_{coh}(\mathcal{D}_{\mathcal{B}_k,\lambda})) \simeq D^b(\mathrm{Mod}_{fg}(U(\mathfrak{g}_k)/U(\mathfrak{g}_k)\,\mathrm{Ker}(c_\lambda)).$$

$(c_\lambda$: (Harish-Chandra center of $U(\mathfrak{g}_k)) \to k)$

Results of Bezrukavnikov-Mirkovic-Rumynin

(center of $\mathcal{D}_{\mathcal{B}_k,\lambda}$) $\doteqdot$ (functions on the twisted cotangent bundle $T^*_\lambda \mathcal{B}_k$).

Hence we have the localization $\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda}$ of $\mathcal{D}_{\mathcal{B}_k,\lambda}$ on $T^*_\lambda \mathcal{B}_k$ so that

$$\{\mathcal{D}_{\mathcal{B}_k,\lambda}\text{-modules}\} \simeq \{\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda}\text{-modules}\}.$$

More precisely, we have

$$\text{(center of } Fr_* \mathcal{D}_{\mathcal{B}_k,\lambda}) = \pi_{\lambda *}(\mathcal{O}_{T^*_\lambda \mathcal{B}_k^{(1)}}),$$

$$Fr : \mathcal{B}_k \to \mathcal{B}_k^{(1)} \qquad \text{(Frobenius twist)},$$

$$\pi_\lambda : T^*_\lambda \mathcal{B}_k^{(1)} \to \mathcal{B}_k^{(1)} \quad \text{(twisted cotangent bundle of } \mathcal{B}_k^{(1)}),$$

$$\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda} := \mathcal{O}_{T^*_\lambda \mathcal{B}_k^{(1)}} \otimes_{\pi_\lambda^{-1}\pi_{\lambda *}(\mathcal{O}_{T^*_\lambda \mathcal{B}_k^{(1)}})} \pi_\lambda^{-1} Fr_* \mathcal{D}_{\mathcal{B}_k,\lambda},$$

$$\text{(localization of } \mathcal{D}_{\mathcal{B}_k,\lambda} \text{ on } T^*_\lambda \mathcal{B}_k^{(1)}).$$

Results of Bezrukavnikov-Mirkovic-Rumynin

Azumaya property

$\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda}$ is an Azumaya algebra of rank $p^{2\dim \mathcal{B}_k}$ on $T^*_\lambda \mathcal{B}_k^{(1)}$, i.e.

- $\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda}$ is a locally free $\mathcal{O}_{T^*_\lambda \mathcal{B}_k^{(1)}}$-module,
- $\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda}(y) \simeq M_{p^{\dim \mathcal{B}_k}}(k) \qquad (y \in T^*_\lambda \mathcal{B}_k^{(1)})$.

moment map $\gamma_\lambda : T^*_\lambda \mathcal{B}_k^{(1)} \to \mathfrak{g}_k^{*(1)}$

Theorem B (BMR)

The restriction $\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda}|\gamma_\lambda^{-1}(\xi)$ is a split Azumaya algebra, i.e.

$$\tilde{\mathcal{D}}_{\mathcal{B}_k,\lambda}|\gamma_\lambda^{-1}(\xi) \simeq \mathcal{E}nd_{\mathcal{O}_{\gamma_\lambda^{-1}(\xi)}}(\mathcal{V})$$

for some locally free $\mathcal{O}_{\gamma_\lambda^{-1}(\xi)}$-module $\mathcal{V}$.

Results of Bezrukavnikov-Mirkovic-Rumynin

$\boxed{\mathfrak{g}_k\text{-module}}$ —— Theorem A —— $\boxed{\mathcal{D}_{\mathcal{B}_k}\text{-module}}$ —— Theorem B —— $\boxed{\mathcal{O}_{T^*\mathcal{B}_k^{(1)}}\text{-module}}$

Corollary (BMR, Lusztig's conjecture)

For $(\lambda, \xi) \in \mathfrak{h}_k^* \times \mathfrak{g}_k^{*(1)}$ such that λ is regular we have

$$
\begin{aligned}
&\sharp\{\text{irreducible } U(\mathfrak{g}_k)\text{-module with} \\
&\qquad \text{Harish-Chandra central character} \longleftrightarrow \lambda, \\
&\qquad\qquad \text{Frobenius central character} \longleftrightarrow \xi\} \\
&= \dim H^*(\gamma_\lambda^{-1}(\xi)).
\end{aligned}
$$

analogue of Theorem B (BMR)

Theorem

$$(\text{center of } Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}) = \pi_{t*}(\mathcal{O}_{V_t}),$$

where

$$V_t = \{(B^- g, (b, b')) \in \mathcal{B} \times K \mid gbb'^{-1}g^{-1} \in t^{2\ell}N^-\},$$
$$K = \{(hn_+, h^{-1}n_-) \in B^+ \times B^- \mid h \in H, n_+ \in N^+, n_- \in N^-\},$$
$$\pi_t : V_t \to \mathcal{B} \quad (\text{projection}).$$

Set

$$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta,t} := \mathcal{O}_{V_t} \otimes_{\pi_t^{-1}\pi_{t*}(\mathcal{O}_{V_t})} \pi_t^{-1} Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t} \qquad (\text{localization of } Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t} \text{ on } V_t).$$

Then

$$\{Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}\text{-modules}\} \simeq \{\tilde{\mathcal{D}}_{\mathcal{B}_\zeta,t}\text{-modules}\}.$$

Azumaya property

Theorem

$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta,t}$ is an Azumaya algebra of rank $\ell^{2\dim\mathcal{B}}$ on V_t.

moment map $\qquad \gamma_t : V_t \to K.$

Theorem B' (analogue of Theorem B (BMR))

The restriction $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta,t}|\gamma_t^{-1}(k)$ is a split Azumaya algebra.

analogue of Theorem A (BMR)

Assume $\ell >$ (Coxeter number) in the following.

Conjecture A'

If $t \in H$ is regular, then

$$D^b(\mathrm{Mod}_{coh}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta,t})) \simeq D^b(\mathrm{Mod}_{fg}(U_\zeta^{DK}(\mathfrak{g})/U_\zeta^{DK}(\mathfrak{g})Ker(c_t)),$$

$$(c_t : \text{(Harish-Chandra center of } U_\zeta^{DK}(\mathfrak{g})) \to \mathbf{C})$$

$U_\zeta^{DK}(\mathfrak{g})$-module — Conjecture A' — $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta,t}$-module — Theorem B' — $\mathcal{O}_{V_t}$-module

Conjecture A' implies the following.

Lusztig's conjecture

For $(t,k) \in H \times K$ such that t is regular we have

$$\begin{aligned}&\sharp\{\text{irreducible } U_\zeta^{DK}(\mathfrak{g})\text{-module with}\\ &\qquad \text{Harish-Chandra central character} \longleftrightarrow t,\\ &\qquad\qquad \text{Frobenius central character} \longleftrightarrow k\}\\ &= \dim H^*(\gamma_t^{-1}(k)).\end{aligned}$$

Conjecture A' is a consequence of the following.

Conjecture I

$$R\Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}) \cong U_\zeta^{DK}(\mathfrak{g}) \otimes_{\mathfrak{z}_{HC}(U_\zeta^{DK}(\mathfrak{g}))} \mathbf{C}_t \qquad (t \in H).$$

Conjecture I is a consequence of the following.

Conjecture II

$$R\Gamma(\mathcal{B}, (Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})_{fin}) \cong (U_\zeta^{DK}(\mathfrak{g}))_{fin} \otimes_{\mathfrak{z}_{HC}(U_\zeta)} \mathbf{C}_t \qquad (t \in H).$$

$(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})_{fin}$ =(ad($U_\zeta^L(\mathfrak{g})$)-finite part of $Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$),

$(U_\zeta^{DK}(\mathfrak{g}))_{fin}$ =(ad($U_\zeta^L(\mathfrak{g})$)-finite part of $U_\zeta^{DK}(\mathfrak{g})$)

Conjecture II is equivalent to the following.

Conjecture III

$$R\,\mathrm{Ind}_\zeta(\mathbf{C}_\zeta[B^-]_{ad}) \cong \mathbf{C}_\zeta[G] \otimes_{\mathbf{C}[H/W]} \mathbf{C}[H]$$

- $\mathrm{Ind}_\zeta : \{\mathbf{C}_\zeta[B^-]\text{-comodule}\} \to \{\mathbf{C}_\zeta[G]\text{-comodule}\}$,
- $\mathbf{C}_\zeta[B^-]_{ad} = (\mathbf{C}_\zeta[B^-]$ with adjoint coaction).

Analogue of Conjecture III for G

$$R\,\mathrm{Ind}_1(\mathbf{C}[B^-]_{ad}) \cong \mathbf{C}[G] \otimes_{\mathbf{C}[H/W]} \mathbf{C}[H]$$
$$\Longleftrightarrow R\Gamma(\mathcal{O}_{G\times^{B^-}(B^-)_{ad}}) \cong \Gamma(\mathcal{O}_{G\times_{H/W}H})$$

- $\mathrm{Ind}_1 : \{\mathbf{C}[B^-]\text{-comodule}\} \to \{\mathbf{C}[G]\text{-comodule}\}$,

We can prove this using a geometric argument.

Reference

T. Tanisaki: Math. Z. 250 (2005), 299—361.
T. Tanisaki: arXiv:1002.0113.
T. Tanisaki: arXiv:1101.5848.

Remark

There is a closely related work using a different definition of D-modules on quantized flag manifolds:

E. Backelin, K. Kremnizer: Adv. Math. Vol 203 (2006), 408–429.
E. Backelin, K. Kremnizer: J. AMS Vol 21 (2008), 1001-1018.