Categorification, Lie algebras and Topology

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Northeastern University

September 21, 2011

This talk is online at

`http://pages.uoregon.edu/bwebster/Banff.pdf`

References:

- Raphaël Rouquier, *2-Kac-Moody algebras*, arXiv:0812.5023.
- Mikhail Khovanov and Aaron Lauda, *A diagrammatic approach to categorification of quantum groups III.*, arXiv:0807.3250.
- Ben Webster, *Knot invariants and higher representation theory I & II*, arXiv:1001.2020 & 1005.4559.

Universal enveloping algebras

Let $\mathfrak{g} = \mathfrak{sl}_n$ (actually any simple/symmetrizable Kac-Moody algebra will do). This is the Lie algebra of trace 0 matrices with the usual commutator.

If we let $E_i = (\delta^i_j \delta^{i+1}_k)_{jk}$, $F_i = E_i^t$ and $H_i = [E_i, F_i]$, then this algebra has a presentation of the form

$$[H_j, E_i] = (2\delta^i_j - \delta^{i+1}_j - \delta^{i-1}_j)E_i \qquad [H_j, F_i] = (-2\delta^i_j + \delta^{i+1}_j + \delta^{i-1}_j)F_i$$

$$[E_i, F_j] = \delta^i_j H_i \qquad [E_i, E_j] = [F_i, F_j] = 0 \qquad (i \neq j \pm 1)$$

$$[E_i, [E_i, E_{i\pm1}]] = [F_i, [F_i, F_{i\pm1}]] = 0$$

By definition, the universal enveloping algebra is the *associative* algebra generated by these symbols subject to the relations above (where $[-,-]$ means commutator).

There's also $\widehat{\mathfrak{sl}}_n$, where indices are taken in $\mathbb{Z}/n\mathbb{Z}$, and relations are the same.

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Universal enveloping algebras

We actually want a slightly bigger algebra $\dot{U}$, with some extra idempotents $\mathbb{1}_\lambda$ for $\lambda \in \mathbb{Z}^{n-1}$. These satisfy the relations

$$\mathbb{1}_\lambda \mathbb{1}_{\lambda'} = \delta^{\lambda}_{\lambda'} \mathbb{1}_\lambda \qquad H_i \mathbb{1}_\lambda = \mathbb{1}_\lambda H_i = \lambda^i \mathbb{1}_\lambda.$$

Note that

$$\mathbb{1}_\lambda E_i = E_i \mathbb{1}_{\lambda-\alpha_i} \qquad \mathbb{1}_\lambda F_i = F_i \mathbb{1}_{\lambda+\alpha_i}$$

where $\alpha_i = (\dots, 0, -1, 2, -1, 0, \dots)$.

We can represent elements of this as pictures on a line

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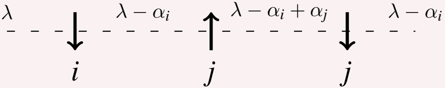

Categorifying UEAs

Chuang and Rouquier and Khovanov and Lauda had the insight that one could make these pictures the objects of a category and showed that one could interpret the morphisms as diagrams in a plane modulo certain relations over a field $\Bbbk$.

The morphisms of $\mathcal{U}$ are given by $\mathbb{C}$-linear combinations of oriented 1-manifolds decorated with dots and labeled with elements of $[1, n-1]$, whose boundaries are the given objects (with orientations and labels), modulo certain relations.

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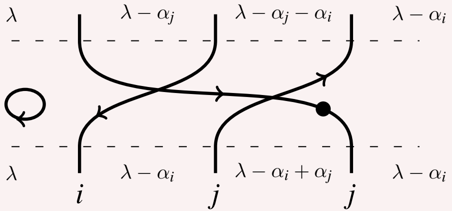
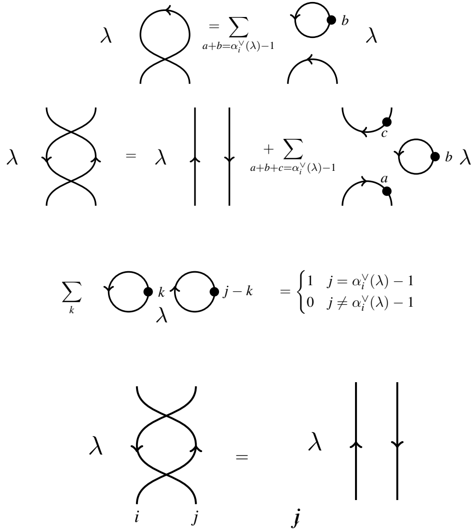
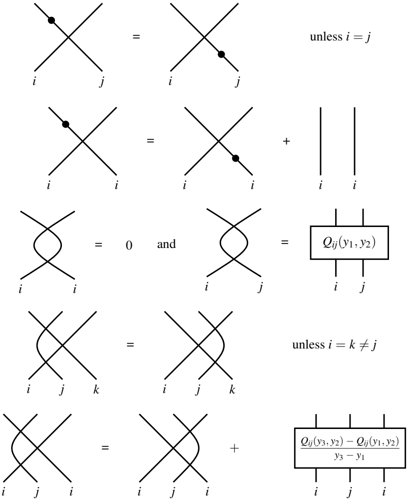


Relations in $\mathcal{U}$

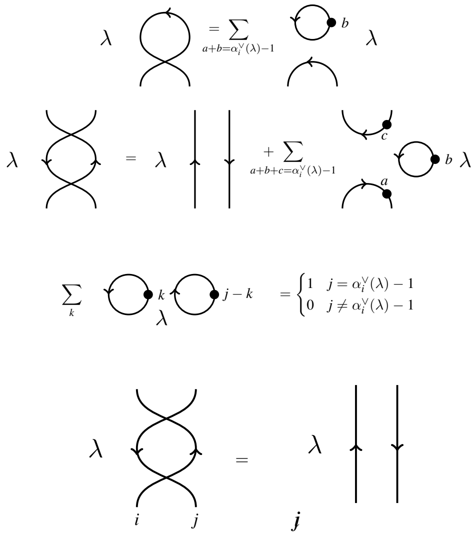


$$Q_{ij}(u,v) = Q_{ji}(v,u) = \begin{cases} 1 & i \neq j \pm 1 \\ au + bv & i = j+1 \end{cases}$$



Relations in $\mathcal{U}$

Note: homogeneous!



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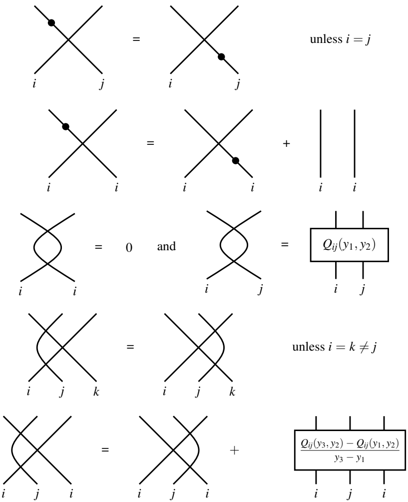

The category $\mathcal{U}$

Those relations may look inscrutable, but actually every single one of them can be guessed by looking at the geometry of quiver varieties.

We let $\mathcal{U}$ be the idempotent completion of the category whose

- objects are diagrams on a line shown above and
- morphisms are $\mathbb{k}$-linear combinations of the pictures in the plane, modulo the relations of the previous slide.

Idempotent completion means adding a new object for each idempotent which is the image of that idempotent as a projection.

Equivalently one can think of the formal sums of diagrams described previously as a big algebra $\mathfrak{U}$. Then $\mathcal{U}$ is just projective modules over that algebra.

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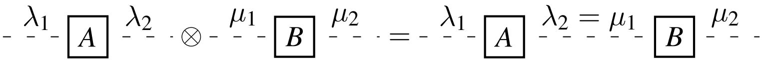
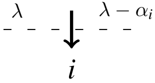
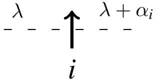
The monoidal structure

The category $\mathcal{U}$ is monoidal; it has a tensor product. Visually, it's quite simple. You just put diagrams next to each other if the label at the edges match, and get 0 if they don't.



The Grothendieck group

Let $\mathcal{E}_i^\lambda =$ 

and $\mathcal{F}_i^\lambda =$ 

Theorem (Khovanov-Lauda, W.)

The GG of $\mathcal{U}$ is $\dot{U}$, via the isomorphism $[\mathcal{E}_i^\lambda] \mapsto \mathbb{1}_\lambda E_i$, $[\mathcal{F}_i^\lambda] \mapsto \mathbb{1}_\lambda F_i$.

For example,

$\mapsto \mathbb{1}_\lambda E_i F_j E_j \mathbb{1}_{\lambda-\alpha_i}$

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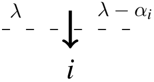
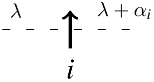
Let $\mathcal{E}_i^\lambda =$ 

and $\mathcal{F}_i^\lambda =$ 

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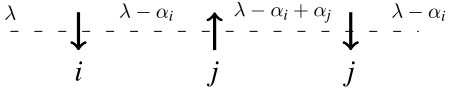
$\Big[$ 

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Grading

As I indicated on the relations slide, the relations are homogeneous for a particular grading; the category $\mathcal{U}$ actually has a graded version $\tilde{\mathcal{U}}$.

Theorem (K.-L.)

The GG of $\tilde{\mathcal{U}}$ is $\dot{U}_q$, the quantized universal enveloping corresponding to $\mathfrak{sl}_n$.

I feel like I've written down enough relations in this talk, so let me take the above as a definition.

As a general rule, it's never harder to work with quantum groups in this picture (sometimes, it even makes things easier); you just pay attention to the grading.

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Representations

One of the reasons people like $\mathfrak{sl}_n$ is that it has a nice representation theory. Every finite dimensional irrep is generated by a unique line killed by all E_i, and the representation V_λ is determined by the weight λ of this line.

So, we can construct a representation $\mathcal{L}_\lambda$ of $\mathcal{U}$ by starting with a single object $\mathbb{V}$ of weight λ with boring endomorphisms, and letting $\mathcal{U}$ act by horizontal composition, subject to $\mathcal{E}_i \otimes \mathbb{V} = 0$.

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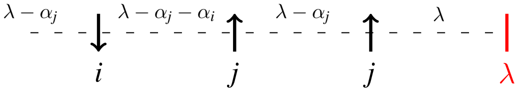

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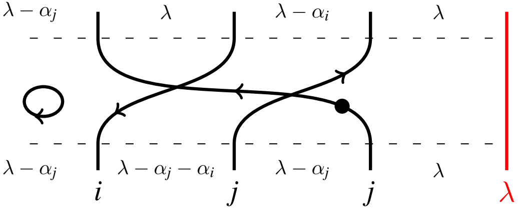
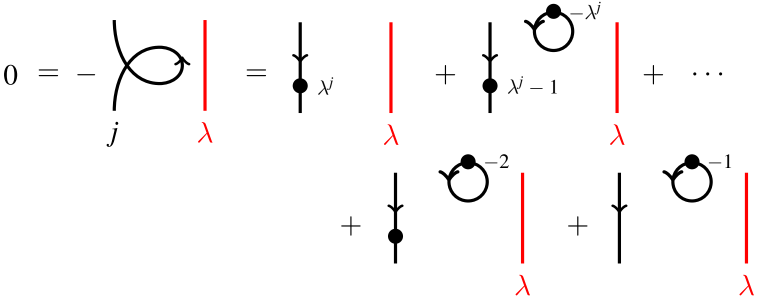


Representations

What are the consequences of this assumption? They are encapsulated in the "cyclotomic relation."



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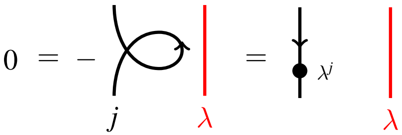

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The GG of $\mathcal{L}_\lambda$ is the irreducible representation of $\dot{U}$ with highest weight λ, and $\mathcal{L}_\lambda$ is essentially the unique such module category for $\mathcal{U}$.

(Small miracle: you might think that this would give you the Verma module; it doesn't!).

One advantage of such a description is that indecomposable modules give a basis of the GG; since $\mathcal{E}_i \otimes -$ or $\mathcal{F}_i \otimes -$ applied to an indecomposable is a sum of indecomposables, E_i, F_i manifestly have positive integral structure coefficients.

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Connections to classical representation theory

You actually know some of these categories:

Theorem (Brundan-Kleshchev)

- *if $\Bbbk$ has characteristic 0, then the representations of all symmetric groups over $\Bbbk$ are the categorification of the Fock space representation of $\mathfrak{sl}_\infty$ (and all fundamental representations for $\mathfrak{sl}_n$ are carefully chosen bits).*
- *if $\Bbbk$ has characteristic p, then the projective modules over all symmetric groups over $\Bbbk$ are the categorification of the basic representation of $\widehat{\mathfrak{sl}}_p$.*
- *if $\Bbbk$ has characteristic 0 and contains a primitive nth root of unity, then the projective representations of all Hecke algebras over $\Bbbk$ with q an nth root of unity are the categorification of the basic representation of $\widehat{\mathfrak{sl}}_n$.*

In all cases, action of $\mathcal{E}_i$ and $\mathcal{F}_i$ are restriction and induction functors.

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Knot invariants

This framework also has applications in topology:

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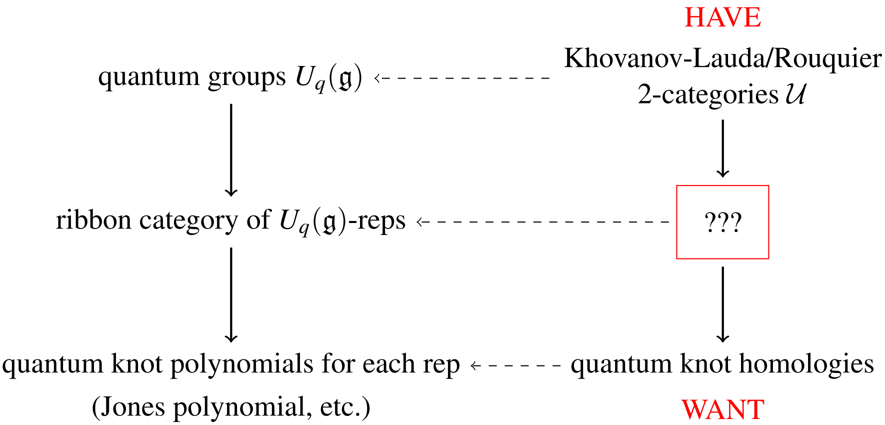

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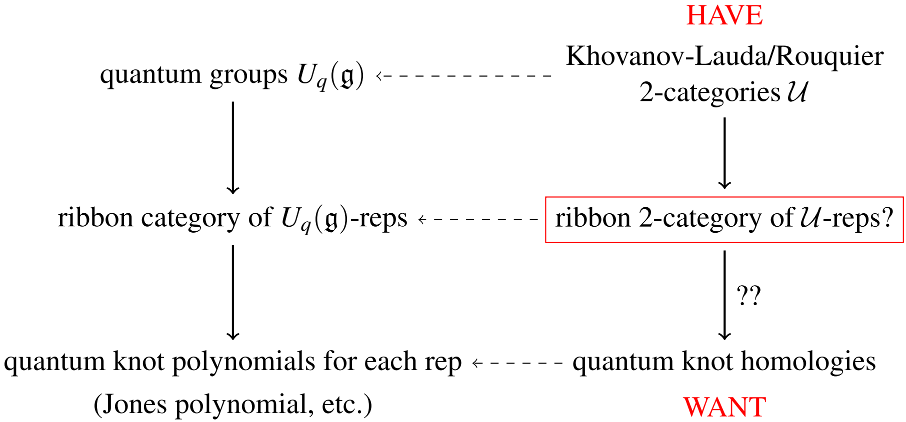

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↓ ↓

ribbon category of $U_q(\mathfrak{g})$-reps ← - - - - - - - - - - categorifications of tensor products of simples

↓ ↓ !

quantum knot polynomials for each rep ← - - - - - - quantum knot homologies

(Jones polynomial, etc.)

HAVE

Tensor products

As with irreducibles or the UEA, we can introduce a graphical calculus for elements of $V_{\underline{\boldsymbol{\lambda}}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$.

- A upward (downward) black line on the left means acting by F_i (E_i).
- A red line at the left labeled by λ corresponds to $v_\lambda \otimes -$, where v_λ is the highest weight vector of V_λ.

So, we obtain a spanning set of $V_{\underline{\boldsymbol{\lambda}}}$ consisting of vectors like

$E_i(v_{\lambda_1} \otimes F_j v_{\lambda_2}) \leftrightarrow$

Exactly as with $\mathcal{U}$ and $\mathcal{L}_\lambda$, we can make these the objects of a category $\mathcal{L}_{\underline{\boldsymbol{\lambda}}}$, with morphisms given by diagrams.

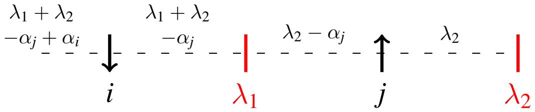
Tensor products

As with irreducibles or the UEA, we can introduce a graphical calculus for elements of $V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$.

- A upward (downward) black line on the left means acting by F_i (E_i).
- A red line at the left labeled by λ corresponds to $v_\lambda \otimes -$, where v_λ is the highest weight vector of V_λ.

So, we obtain a spanning set of $V_{\underline{\lambda}}$ consisting of vectors like

$E_i(v_{\lambda_1} \otimes F_j v_{\lambda_2}) \leftrightarrow$



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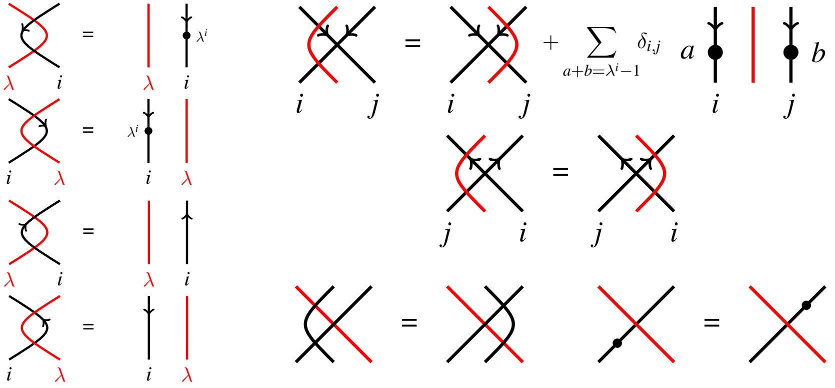
Tensor products

We should have some more relations (note: also homogeneous).



Tensor products

Theorem (W.)

The GG of $\mathcal{L}_{\underline{\boldsymbol{\lambda}}}$ is $V_{\underline{\boldsymbol{\lambda}}}$. When the Cartan matrix is symmetric and $\Bbbk$ has characteristic 0, the classes of indecomposables give Lusztig's canonical basis of $V_{\underline{\boldsymbol{\lambda}}}$.

For ADE, it immediately follows that Lusztig's canonical basis of $\dot{U}$ is given by the indecomposables of $\mathcal{U}$.

For non-symmetric type or positive characteristic, you get a new basis, which I call **orthodox** on which totally non-negative elements act by non-negative structure coefficients.

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Classical representation theory again

For a tensor product of fundamental weights:

- If $\Bbbk$ has characteristic 0, then $\mathcal{L}_{\underline{\lambda}}$ for $\mathfrak{sl}_n$ is a sum of blocks of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_k$, with k and the parabolic fixed by the choice of $\underline{\lambda}$.
- $\mathcal{L}_{\underline{\lambda}}$ for $\widehat{\mathfrak{sl}}_n$ is the category of some projective modules over a cyclotomic q-Schur algebra (those corresponding to multipartitions whose constituents are all n-regular).
- If one applies this schema to $\widehat{\mathfrak{gl}}_n$, then $\mathcal{L}'_{\underline{\lambda}}$ is all projective modules over a cyclotomic q-Schur algebra.

All these realizations of classical categories in terms of higher representation theory have another very interesting consequence: all the previously mentioned categories have hidden gradings!

In particular, all their “decomposition numbers” have q-analogues, which are often the coefficients of a canonical basis in terms of a standard one (in the representation they categorify).

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Decomposition numbers

Theorem

- *The graded decomposition numbers of parabolic category $\mathcal{O}$ (Kazhdan-Lusztig polynomials) are exactly the coefficients of the canonical basis of tensor product of miniscule $\mathfrak{sl}_n$ representations in terms of the standard basis.*
- *The graded decomposition numbers of a Hecke algebra at an nth root of unity are exactly the coefficients of the canonical basis of the basic representation of $U_q(\widehat{\mathfrak{sl}}_n)$ in terms of the regular wedge basis.*
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Geometry and algebra

What you've seen thus far is what I'd call the "combinatorial" perspective on these algebras, but they have another face. Assume for now that $\mathfrak{g}$ is symmetric.

Given a sequence of weights $\underline{\boldsymbol{\lambda}}$, we have an associated quiver variety with $\mathbb{C}^*$-action. This quiver variety is symplectic and has a natural universal deformation quantization A; the action of $\mathbb{C}^*$ extends to an inner action on A.

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$\mathcal{L}_{\underline{\lambda}}$ is derived equivalent to any generic block of the associated category $\mathcal{O}$ for A, the category of finitely generated A-modules which are locally finite for the non-negative weight subalgebra $A^{\geq 0}$.

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Symplectic reflection algebras

One particularly interesting case is that of the spherical symplectic reflection algebras of $S_n \wr \Gamma$ which quantize quiver varieties of the affine Dynkin diagram associated to Γ by McKay with special dimension vectors.

Theorem

The category of finite dimensional representations of the SRA for $S_n \wr \Gamma$ for any integral spherical parameter is derived equivalent to the $\Lambda_0 - n\delta$ weight space of $\mathcal{L}^{\Gamma}_{\Lambda_0}$.

In particular, its K-group is the associated weight space, with Euler form the Shapovalov form, and for some parameters, the simples are Lusztig's canonical basis.

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When $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$, the SRA is a Cherednik algebra, and has a category $\mathcal{O}_{\mathbf{c}}$.

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For any integral spherical parameter, $\mathcal{O}_{\mathbf{c}}$ for $S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ is derived equivalent to representations of a rank n q-Schur algebra, where q is a primitive ℓth root of unity.

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Thanks for listening.