Endomorphism Rings

Section One: Endotrivial modules

These are kG-modules V such that $\text{End}_k(V) \cong k \oplus P$, where P is a projective kG-module, and were introduced by E.C. Dade. They are a class of modules which appear tractable in the vast menagerie of (mostly wild) modules. They also play an important role when studying the multiplicative structure of representation rings of various types (since they are "close to invertible"). Recently, G. Navarro and I proved:

Theorem: *Let G be a finite p-solvable group with an elementary Abelian subgroup of order p^2. Then a simple kG-module M is endotrivial if and only if* $\dim_k(M) = 1$.
This completed the classification by Carlson, Mazza and Thévénaz of the torsion part of the group of endotrivial modules for such G. However, our proof does require CFSG, specifically properties of the automorphism groups of known simple groups.

For general G, I recently proved that for a general finite group G containing an elementary Abelian subgroup of order p^2, and possessing a faithful simple module V which is endotrivial, either $F^*(G) = Z(G)L$ where L is quasisimple and irreducible on V or else V is induced from a one dimensional representation of a strongly p-embedded subgroup of G. Thus the classification of simple endotrivial modules (in rank greater than 1) reduces to questions about simple endotrivial modules for quasi-simple groups.

Some remarks on Virtually irreducible lattices

Now let R be a complete discrete valuation ring with ring with unique maximal ideal $\pi = J(R)$. An indecomposable RG module V is said to be *virtually irreducible* if $\nu_\pi(\text{trace}(\alpha)) \geq \nu_\pi(\text{Rank}(V))$, for all $\alpha \in \text{End}_{RG}(V)$, with equality if and only if α is invertible. These modules were introduced by R. Knörr in his study of Brauer's height zero conjecture. In particular, such module for Abelian p-groups are of interest in this context. A source for an absolutely irreducible RG-lattice is virtually irreducible. The question then is: if B is a block of RG with Abelian defect group D, what restrictions can be put on the rank of a virtually irreducible RD-module V. The height of V is $\frac{\nu_\pi(\text{rank}(V))}{\nu_\pi(p)}$. M. Geline has recently proved that height$(V) < \frac{d}{2}$ if D is elementary Abelian. However, we have:

Theorem (M. Geline and GRR): *For each positive integer h and prime p, any elementary Abelian p-group D of order p^{2h+2} has a virtually irreducible RD-module of height h which affords a multiplicity-free character.*

The proof is instructive, and perhaps a little unusual, so I would like to say a few words about it. A virtually irreducible RD-module V has height h if and only if $\text{rank}_R(V)$ is divisible by p^h and $\chi(x) \equiv \text{rank}_R(V) \pmod{\pi p^h}$ for all $x \in D$. We are looking for a multiplicity free character θ of D such that $\nu_\pi(|\theta(x)|^2) = 2h\nu_\pi(p)$ for all $x \in D$. This is a little reminiscent of (v, k, λ)-difference sets. A (v, k, λ) difference set on an Abelian group G of order v is a subset S of G such that $S^+(S^+)^o = (k - \lambda)1_G + \lambda G^+$, where the superscript $+$ denotes the sum in the group algebra $\mathbb{C}G$ and o is the polarity induced by inversion. Note that S has cardinality k and that we have $k^2 - k = \lambda(v - 1)$.

It is well known (really an observation dating back to Turyn) that a subset S of cardinality k is a (v,k,λ)-difference set if and only if we have $|\mu(S^{+})|^2 = k - \lambda$ for each non-trivial character μ of G. We prefer to take a "dual" viewpoint: if we consider the Abelian group G as the group of linear characters of its dual group (ie group of linear characters) $\widehat{G}$, then a difference set S corresponds precisely to a multiplicity free character of degree k of $\widehat{G}$ whose absolute value squared takes constant value $k - \lambda$ on each non-identity element of $\widehat{G}$. Since the groups G and $\widehat{G}$ are isomorphic, the search for (v,k,λ)-difference sets is nothing other than the search for multiplicity free characters of degree k of G whose absolute value squared takes constant value $k - \lambda$ on $G^{\#}$.

When D is an elementary Abelian p-group of order p^{2h+2}, the search for multiplicity free characters θ such that $|\theta(g)|^2$ is divisible by p^{2h}, but lies outside πp^{2h} for each $g \in D^{\#}$ is rather less demanding that the search for a difference set.

Notice that $\hat{D}$ has $p^{h+1}+1$ subgroups of order p^{h+1} which intersect pairwise in the identity (to see this, note that a two-dimensional vector space over GF(q) is a union of $q+1$ 1-dimensional subspaces). Let $\widehat{S}$ be the union of the non-identity elements of any p^h of these subgroups, say $\widehat{S} = \cup_{j=1}^{p^h} \hat{D}_j^{\#}$. Notice that $\hat{D}_k\hat{D}_\ell = \hat{D}$ whenever $k \neq \ell$. Let μ be a non-trivial linear character of $\hat{D}$. Then there is at most one value of j with $\hat{D}_j \leq \ker\mu$. If there is no such value of j, then we have $\mu(\widehat{S}) = -p^h$, while if there is one such value of j, we have $\mu(\widehat{S}) = (p^{h+1}-1)-(p^h-1) = p^{h+1}-p^h$. Hence $\widehat{S}$ is the required multiplicity free character of D, and is a virtually irreducible character of height h. Only when $p=2$ is this a genuine difference set, and it is then one constructed by J.F. Dillon. In fact, the lattice of the theorem comes from a $\mathbb{Z}G$-module.

Lifting a representation of M_{11}

The Mathieu group has a 5-dimensional representation over GF(3) (in fact, it has two such representations, which are not dual to each other). This representation does not "lift" to a complex representation. However, the group M_{11} is a homomorphic image of the amalgam $G = \mathrm{GL}(2,3) *_{D_8} S_4$, and the 5-dimensional representation characteristic 3 representation of M_{11} may be viewed as a representation of G. As such, it may be liftable to a complex representation of G. The challenge is to find compatible representations of $\mathrm{GL}(2,3)$ and S_4 (which agree on the intersection D_8). Expressing G as such an amalgam comes from considering its fusion system, which is maximal for s semidihedral 2-group of order 16.

Let

$$a = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$c = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{\sqrt{-2}} & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{-2}} & 0 & 0 \\ \frac{1}{\sqrt{-2}} & \frac{1}{\sqrt{-2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1-\sqrt{-2}}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{-1+\sqrt{-2}}{2} \end{pmatrix},$$

$$d = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{2} & \frac{-1-\sqrt{-2}}{2} & 0 & 0 \\ 0 & \frac{1-\sqrt{-2}}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $H = \langle b, c\rangle \cong$ GL(2, 3) and $K = \langle a, b, d\rangle \cong S_4$. Then $\langle H, K\rangle$ is a homomorphic image of G, and we have constructed a representation of G via a subgroup of SU(5, $\mathbb{Z}[\frac{1}{\sqrt{-2}}]$). Note also that this representation allows reduction (mod p) for all odd primes p. A GAP computation shows that the reduction (mod 3) of this representation leads to a representation of M_{11} over GF(3). In fact, there are two choices, since we could take $\sqrt{-2}$ to be 1 or -1. Hence two algebraically conjugate representations of G lead to the code and cocode representation of M_{11}, which I find interesting.

However, when this representation of G is reduced modulo other odd primes, nothing out of the ordinary happens (but something into the modular does). The reduction (mod p) leads to either PSL(5, p) for $p \equiv 1, 3 \pmod 8$ (and $p > 3$), or PSU(5, p) for $p \equiv 5, 7 \pmod 8$. It is also interesting to note that the amalgam $G \cong$ SU(3, $\mathbb{Z}[\frac{1}{\sqrt{-2}}]$), which has been proved by J-P. Serre.