Lie powers for general linear groups and Lie modules for symmetric groups

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$F =$ infinite field, char$(F) = p$,
$V = F$-vector space, $\dim V = r$.

(I) $L(V) =$ free Lie algebra, $= \bigoplus_{n \geq 1} L^n(V)$

$$L^n(V) \;\cong\; (V^{\otimes n})\omega_n$$

$\omega_n = (1 - c_2)(1 - c_3) \ldots (1 - c_n)$,
$c_k = (k \;\; k-1 \;\; \ldots \;\; 2 \;\; 1)$

(II) $\text{Lie}(n) = F\mathcal{S}_n\omega_n$, ($\mathcal{S}_n$ symmetric group).

$\text{Lie(n)} \cong \text{fL}^{\text{n}}\text{(V)}$ when $r \geq n$.

$\dim \text{Lie(n)} = \text{(n} - 1\text{)!}$.

(III) $M(V) =$ free metabelian Lie algebra,
$M(V) = \bigoplus_{d \geq 0} M^d(V)$

$$M^d(V) \cong \nabla(d - 1, 1)$$

Tilting summands

If $p \nmid n \;\Rightarrow\; L^n(V)$ is a direct summand of $V^{\otimes n}$.

$$\Rightarrow \quad L^n(V) \cong \bigoplus_\lambda m_\lambda T(\lambda)$$

If $p|n$ and $r \geq n$ then $L^n(V)$ still has 'tilting summands'.

Qu Which $T(\lambda)$ occur?

[Klyachko'74] char$(F) = 0$:
If $n \neq 4, 6$ and $\lambda \neq (n), (1^n)$ then $L(\lambda)$ occurs in $L^n(V)$.

[R. Bryant, M. Johnson, 2010]
Almost all $T(\lambda)$ occur as summands in $L^n(V)$ if $n \neq p^m$, or $2p^m$.

[M. Johnson, E, 2010] Case $r = 2$, all n (≤ 3 partitions unknown).

Block components

$\text{Lie(n)} = \bigoplus_{\text{B block}} \text{Lie(n)}_{\text{B}}$

Blocks of $F\mathcal{S}_n$ are parametrized by p-cores. The principal block has p-core $\emptyset$.

Theorem [E-Tan] *Any non-projective indecomposable summand of* Lie(n) *belongs to the principal block of* $F\mathcal{S}_n$.

Theorem [E-Tan] *Any indecomposable summand of* $L^n(V)$ *not of form* $T(\lambda)$ *for* λ *p-regular, belongs to the principal block (class) of the Schur algebra* $S(r,n)$.

Structure of non-principal block component

Let $n = p^m k$ where $p \nmid k$. Let B be a block of $F\mathcal{S}_n$ with core κ and $\kappa \neq \emptyset$.

Theorem [R. Bryant, E] *Let $\mathcal{S}_k^* \leq \mathcal{S}_n$ be the diagonal in the Young subgroup $(\mathcal{S}_k)^{p^m}$. View* Lie(k) *as $\mathcal{S}_k^*$-module. Then*

$$\mathrm{Lie(n)_B} \cong \frac{1}{\mathrm{p^m}}(\mathrm{Lie(k)} \uparrow_{\mathcal{S}_\mathrm{k}^*}^{\mathcal{S}_\mathrm{r}})_\mathrm{B}$$

Special case $r = p^m$.

Theorem *Let* $\mathsf{Lie(n)_B} = \bigoplus_\lambda \mathsf{m}_\lambda \mathsf{P}(\lambda)$. *Then*

$$m_\lambda = \frac{1}{n}\sum_{d|k}\mu(d)\beta^\lambda(\tau^{k/d})$$

where τ *has cycle type* (k^{p^m}). *Here* β^λ *is the Brauer character of the simple module* D^λ *for* $\mathcal{S}_n$.

Case $k = n$: Recovers formula [Donkin-E].

Case char$(F) = 0$: Recovers classical formula [Wever].

Analog for $L^n(V)$:

$$L^n(V)_B \cong \frac{1}{p^m} L^k(V^{\otimes p^m})_B$$

$L^n(V)_B \cong \bigoplus_\lambda m_\lambda T(\lambda)$ with m_λ as above.

Reduction to prime power degree

Let $n = p^m k$ where $p \nmid k$ and $k > 1$.

Theorem [R. Bryant, M. Schocker]

$$L^n(V) \cong \bigoplus_{n=tq, q=p^j} L^q(B_t)$$

where B_t is a direct summand of $V^{\otimes t}$ (independent of n).

Translating to Lie(n):

Theorem [Lim, Tan]

$$\mathrm{Lie(n)} \cong \bigoplus_{\mathrm{n=tq,q=p^j}} (\mathrm{X_t^{\otimes q}} \otimes \mathrm{Lie(q)}) \uparrow_{\mathcal{S}_\mathrm{t} \wr \mathcal{S}_\mathrm{q}}^{\mathcal{S}_\mathrm{n}}$$

where $X_t = fB_t$ when $\dim V \geq n$

Some applications, eg:

Theorem [E, Lim, Tan] *Let* $cx(M)$ *be the complexity of the module* M*. Then* $cx(\mathsf{Lie(n)})$ *is the maximum of* $\{cx(\mathsf{Lie(p^j)}) : 1 \leq \mathsf{j} \neq \mathsf{m}\}$*. Moreover*

$$cx(\mathsf{Lie(p^m)}) \leq \mathsf{m}.$$

Lie(p^m)

Lie(n) = Lie(n)^{pf} $\oplus$ Lie(n)^{proj} where Lie(n)^{pf} has no projective summand, and Lie(n)^{proj} is projective.

Lie(p)^{pf} $\cong \Omega$(F).

Lie(4) $\cong \Omega^{-1}$(D) where $D = D^{(3,1)}$.

[S. Danz, J. Müller] Computer calculations & ingenious guess.

- Lie(8)^{pf} is indecomposable, with vertex a regular C_2^3. The source is an endo-permutation module (EPM), a la Dade.

- Lie(9)^{pf} is indecomposable, vertex a regular C_3^2. The source is an EPM. [Sources explicitly]

Question Is Lie(p^{m})$^{\mathsf{pf}}$ always indecomposable? Is a vertex a regular C_p^m? Is the source EPM?

Question What is the analog for $GL(V)$ of an EPM?

The module M^d

This can be defined by

$$0 \to M^d \to V \otimes S^{d-1} \to S^d \to 0$$

$S^d \cong \nabla(d)$ and $M^d \cong \nabla(d-1,1)$.

[Doty, Krop, et al] Submodule lattice of S^d.

DEF A p-good filtration of a module is a a filtration with quotients isomorphic to $L(\lambda) \otimes \nabla(\mu)^F$ (various λ, μ with λ p-restricted).

S^d has a p-good filtration.

[E, L. Kovacs] Construct a p-good filtration of M^d explicitly.

Determine CF's. Show M^d is multiplicity-free.

RKs (1) [Bakhturin, in Russian] description of the complete submodule lattice, in the language of varieties.

(2) No condition on prime needed.

(3) Generalizing to $\nabla(d-t, 1^t)$ feasible.

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