Extensions of tempered modules

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- (with Solleveld) For U, V tempered irreducible modules: Explicit computation of $\mathrm{Ext}^i_{\mathcal{S}}(U, V)$.

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- Put $R = (R_0^\vee)^{(1)} = \{a = \alpha^\vee + k \mid \alpha \in R_0,\ k \in \mathbb{Z}\}$.
- Write $S_0 = \{s_1, \ldots, s_n\}$ and $F = \{a_0, a_1 \ldots, a_n\} \subset R$ (fundamental affine roots).

Definition of affine Hecke algebra

Choose indeterminates v_s for $s \in S$ with $v_s = v_{s'}$ if $s \sim_W s'$. Put $\Lambda = \mathbb{C}[v_s^{\pm 1} \mid s \in S]$ (base ring). The affine Hecke algebra $\mathcal{H}_\Lambda$ is the unital associative free Λ-algebra with basis T_w $(w \in W)$ subject to the relations:

- If $u, v \in W$ and $l(uv) = l(u) + l(v)$ then $T_u T_v = T_{uv}$.
- For all $s \in S$: $(T_s - v_s)(T_s + v_s^{-1}) = 0$.

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- Given $q(s)^{1/2} \in \mathbb{C}^\times$ such that $q(s)^{1/2} = q(s')^{1/2}$ if $s \sim_W s'$, we write (abusively) $\mathcal{H} = \mathcal{H}(W, q) := \mathcal{H}_\Lambda(W) \otimes_\Lambda \mathbb{C}_{q^{1/2}}$.

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- Denote by $(C_*(\Sigma), \partial_*)$ be the corresponding augmented chain complex.

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- Define $P_k(V) \subset \bigoplus_{J \subset F, |J| = n-k} \mathcal{H} \otimes_{\mathcal{H}(W_J, q)} V \otimes C_k(\Sigma)$ by

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- Define $\epsilon(J, J') \in \{0, \pm 1\}$ by $\partial f_J = \sum_{J'} \epsilon(J, J') f_{J'}$. Define $d_k : P_k(V) \to P_{k-1}(V)$ for $k > 0$ by

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and define $d_0 : P_0(V) \to P_{-1}(V) \simeq V$ by:

$$d_0(h \otimes_{\mathcal{H}(W_J,q) v} \otimes f_J) = (\text{orientation}(f_J))\pi(h)v \in V$$

Theorem (O., Reeder, Solleveld)

Let $(\pi, V) \in \mathrm{Mod}(\mathcal{H})$.

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Proof.

Define (with $J \subset F$ such that $|J| = n - k$, and $w \in W$)

$$\phi_k : C_k(\Sigma) \otimes V \xrightarrow{\sim} P_k(V)$$
$$w(f_J) \otimes v \to T_w \otimes_{\mathcal{H}(W_J,q)} \pi(T_w^{-1})v \otimes f_J$$

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Now check that ϕ_* is an isomorphism of chain complexes from $(C_*(V) \otimes V, \partial_* \otimes \mathrm{id}_V)$ to $(P_*(V), d_*)$. Finally, if $\mathcal{H}(W_J, q)$ is semisimple for $J \neq F$ then $P_k(V)$ is projective for $k \geq 0$. □

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Corollary

$(P_*(\mathcal{H}), d_*)$ *is a projective resolution of $\mathcal{H}$ as $\mathcal{H} \otimes \mathcal{H}^{op}$-module.*

Let $q(s) > 0$ for all $s \in S$ from now on.

Schwartz algebra completion $\mathcal{S}$ of $\mathcal{H}$

Define $\mathcal{S} = \{s = \sum_{w \in W} c_w T_w \in \mathcal{H}^* \mid \forall n \in \mathbb{N} : \; p_n(s) := \sup_{w \in W}\{|c_w|(1 + l(w))^n\} < \infty\}$.

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Theorem

$\mathcal{S}$ is a nuclear Fréchet algebra.

The structure of $\mathcal{S}$ is well understood via the Fourier transform:

Theorem (with Delorme)

$$\mathcal{S} \cong \bigoplus_{(P,\delta)/\sim} C^\infty(T_u^P, \mathrm{End}(\mathcal{V}_{(P,\delta)}))^{W_{(P,\delta)}}$$

- $P \subset F_0$ runs over the subsets of F_0.
- T_u^P the group of (unitary) characters of the central subalgebra $\mathbb{C}[X^P]$ of $\mathcal{H}^P$ of the "Levi subalgebra" $\mathcal{H}^P \subset \mathcal{H}$.
- (V_δ, δ) is a discrete series module over the semisimple quotient $\mathcal{H}_P$ of $\mathcal{H}^P$.
- $\mathcal{V}_{(P,\delta)}$ is a trivial vector bundle, with fiber at $t \in T_u^P$ given by $\mathrm{Ind}_{\mathcal{H}^P}^{\mathcal{H}}(V_{\delta_t})$. Here δ_t is the twist of δ by t.
- $W_{(P,\delta)}$ is a finite group acting projectively on $\mathcal{V}_{(P,\delta)}$ via (unitary) intertwiners.

Goal and motivation

We want to extend the results on projective resolutions of $\mathcal{H}$ to Fréchet modules over $\mathcal{S}$. We intend to use the detailed structural information of $\mathcal{S}$ compute Ext. This gives a useful interplay between homological algebra and harmonic analysis.

Example

Let (U, δ) be a discrete series module of $\mathcal{H}$. Then (U, δ) extends to a projective $\mathcal{S}$-module. Hence for all tempered $\mathcal{H}$-modules (V, π) we should have $\mathrm{Ext}^i_{\mathcal{S}}(U, V) = 0$ for all $i > 0$.

There are some difficulties to overcome.

- $\mathcal{H} \subset \mathcal{S}$ is not a flat extension.
- How to define $\mathrm{Ext}^i_{\mathcal{S}}(U, V)$ for $U, V \in \mathrm{Mod}_{Fré}(\mathcal{S})$? As usual, categories of topological modules are not abelian since images of continuous maps are not necessarily closed.
- Topologically free Fréchet modules $U := \mathcal{S} \hat{\otimes} F$ (where F is a Fréchet space, and $\hat{\otimes}$ stands for the projective completed tensor product) are not necessarily projective in $\mathrm{Mod}_{Fré}(\mathcal{S})$, since subspaces are not necessarily complemented.

Solution (Mac Lane, Connes)

Only work with admissible exact sequences, i.e. exact sequences where all kernels are complemented (as subspaces). The category $\mathrm{Mod}_{Fré}(\mathcal{S})$ with the collection $\mathcal{E}$ of admissible short exact sequences is exact in the sense of Quillen.

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Corollary (Quillen)

There exists an abelian category $\mathcal{A}$ and an equivalence $G : \mathrm{Mod}_{Fré}(\mathcal{S}) \to \mathcal{M}$ onto a full subcategory $\mathcal{M}$ of $\mathcal{A}$ which is closed for extensions, such that $G\mathcal{E}$ consists of the short exact sequences in $\mathcal{A}$ whose objects are in $\mathcal{M}$.

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Definition of Ext

One now defines $\mathrm{Ext}^i_{\mathcal{S}}(U, V)$ using the abelian category $\mathcal{A}$.

Let $V \in \mathrm{Mod}_{Fré}(\mathcal{S})$; let $d_k : P^t_k(V) \to P^t_{k+1}(V)$ as before with

$$P^t_k(V) = \bigoplus_{J \subset F, |J| = n-k} \mathcal{S} \hat{\otimes}_{\mathcal{H}(W_J, q)} V \otimes \mathbb{C} f_J$$

Theorem

$(P^t_k(V), d_*)$ is an admissible projective resolution in $\mathrm{Mod}_{Fré}(\mathcal{S})$.

Proof.

Let $\gamma_k : C_k(\Sigma) \to C_{k+1}(\Sigma)$ be a contraction, and define $\tilde{\gamma}_k$ by:

$$\begin{array}{ccc} C_k(\Sigma) \otimes V & \xrightarrow[\phi_k]{\sim} & P_k(V) \\ \downarrow{\scriptstyle \gamma_k \otimes \mathrm{id}_V} & & \downarrow{\scriptstyle \tilde{\gamma}_k} \\ C_{k+1}(\Sigma) \otimes V & \xrightarrow[\phi_{k+1}]{\sim} & P_{k+1}(V) \end{array}$$

Can choose γ_k so that $\tilde{\gamma}_k$ extends continuously to $P^t_k(V)$. □

Corollary (global dimension $\mathrm{Mod}_{Fr\acute{e}}(\mathcal{S})$)

The global dimension of the exact category $\mathrm{Mod}_{Fr\acute{e}}(\mathcal{S})$ *is n.*

Theorem (Comparison Theorem)

Let U, V be finite dimensional tempered $\mathcal{H}$-modules. Then for all i we have:

$$\mathrm{Ext}^i_{\mathcal{H}}(U, V) \simeq \mathrm{Ext}^i_{\mathcal{S}}(U, V)$$

Proof.

The complexes $\mathrm{Hom}_{\mathcal{H}}(P_*(U), V)$ and $\mathrm{Hom}_{\mathcal{S}}(P^t_*(U), V)$ are equal. □

The comparison theorem implies that if U is discrete series then $\mathrm{Ext}^i_{\mathcal{H}}(U, V) = 0$ for all $i > 0$. Let us be more ambitions and compute Ext between arbitrary irreducible tempered modules using the comparison theorem. Recall the structure theorem

$$\mathcal{S} \cong \bigoplus_{(P,\delta)/\sim} C^\infty(T_u^P, \mathrm{End}(\mathcal{V}_{(P,\delta)}))^{W_{(P,\delta)}}$$

Let $t \in T_u^P$ and let ξ denote the triple $\xi = (P, \delta, t)$. Denote by $V_\xi := Ind_{\mathcal{H}^P}^{H}(V_{\delta_t})$ the induced tempered module of $\mathcal{H}$ which is the fiber of the vector bundle $\mathcal{V}_{(P,\delta)}$ at $t \in T_u^P$. Following Harish-Chandra, Knapp and Stein, Silberger, one proves:

Theorem (O.-Delorme)

- *Let $W_\xi \subset W_{(P,\delta)}$ be the isotropy subgroup of $\xi = (P, \delta, t)$. There exists a canonical decomposition $W_\xi = W(\xi) \rtimes R_\xi$ where $W(\xi)$ is a real reflection group acting on the tangent space T_ξ of T_u^P at t, and R_ξ a group of outer automorphisms of W_ξ.*

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- *The normalized intertwining operators* $I_w \in \mathrm{End}_{\mathcal{H}}(V_\xi)$ *with* $w \in W(\xi)$ *act by scalar multiplications.*

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- *The normalized intertwining operators* $I_w \in \mathrm{End}_{\mathcal{H}}(V_\xi)$ *with* $w \in W(\xi)$ *act by scalar multiplications.*
- *The standard intertwining operators define an isomorphism* $I : \mathbb{C}[R_\xi, \kappa_\xi] \to \mathrm{End}_{\mathcal{H}}(V_\xi)$*, where* κ_ξ *is the 2-cocycle of* R_ξ *defined by projective action of the normalized intertwiners.*

Theorem (Extended Knapp-Stein theorem, O.-Solleveld)

Let $m_\xi \in P(T_\xi)^{W(\xi)}$ be the ideal of $W(\xi)$-invariant polynomials on the tangent space T_ξ, vanishing at ξ. Put $E_\xi = m_\xi / m_\xi^2$, a real representation of R_ξ. Let R_ξ^* be a Schur-extension of R_ξ and let $p \in \mathbb{C}[R_\xi^*]$ be the central idempotent such that $\mathbb{C}[R_\xi, \kappa_\xi] = p(\mathbb{C}[R_\xi^*])$. Let $\widehat{\mathcal{Z}(\mathcal{S})}_{W_{(P,\delta)}\xi}$ denote the formal completion of the center $\mathcal{Z}(\mathcal{S})$ of $\mathcal{S}$ at the central character $W_{(P,\delta)}\xi$. Then

- $\widehat{\mathcal{Z}(\mathcal{S})}_{W_{(P,\delta)}\xi} \simeq \widehat{S(E_\xi)}_\xi^{R_\xi}$.

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- $\widehat{\mathcal{Z}(\mathcal{S})}_{W_{(P,\delta)}\xi} \simeq \widehat{S(E_\xi)}_\xi^{R_\xi}$.
- *The formal completion* $\widehat{\mathcal{S}}_{W_{(P,\delta)}\xi} := \widehat{\mathcal{Z}(\mathcal{S})}_{W_{(P,\delta)}\xi} \otimes_{\mathcal{Z}(S)} \mathcal{S}$ *is Morita equivalent to the ring* $p^*(\widehat{S(E_\xi)} \rtimes R_\xi^*)$.

Theorem (O.-Solleveld)

Denote by Φ_ξ the Morita equivalence from $\mathrm{Mod}^{fd}(\widehat{\mathcal{S}}_{W_{(P,\delta)}\xi})$ *to* $\mathrm{Mod}^{fd}(p^*(\widehat{S(E_\xi)}\rtimes R_\xi^*))$. *Let π, π' be irreducible modules over $\mathcal{S}$. If they have distinct central characters for the center $\mathcal{Z}(\mathcal{S})$ of $\mathcal{S}$ then* $\mathrm{Ext}^i_{\mathcal{H}}(\pi,\pi') = 0$ *for all $i \in \mathbb{Z}$. If both π, π' have central character $W_{(P,\delta)}\xi$, then for all $i \in \mathbb{Z}$ we have*

$$\mathrm{Ext}^i_{\mathcal{H}}(\pi,\pi') \simeq (\Phi_\xi(\pi)^* \otimes \Phi_\xi(\pi') \otimes {\textstyle\bigwedge}^i(E_\xi^*))^{R_\xi}$$

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$\mathrm{Ext}^i_{\mathcal{H}}(\pi, \pi') \simeq (\Phi_\xi(\pi)^* \otimes \Phi_\xi(\pi') \otimes \bigwedge^i (E_\xi^*))^{R_\xi}$

About the proof. The outline is clear: We apply the comparison theorem and then we would like to apply the formal completion functor $\widehat{\mathcal{Z}(\mathcal{S})}_{W_{(P,\delta)}\xi} \hat{\otimes}_{\mathcal{Z}(\mathcal{S})}$ to the projective resolution $P^t(\pi)$ of π as $\mathcal{S}$-module in order to change to the base ring to $\widehat{\mathcal{S}}_{W_{(P,\delta)}\xi}$. Finally we apply the Morita equivalence and use Koszul resolutions to compute the Ext-groups for the cross product ring $\widehat{S(E_\xi)} \rtimes R_\xi^*$.

There are various technical difficulties to overcome, however:

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- Show that formal completion is still exact in this non Noetherian context.
- Formal completion does not preserve continuous linear splittings.
- One has to make of $\mathrm{Mod}_{Fré}(\mathcal{S})$ and $\mathrm{Mod}_{Fré}(\widehat{\mathcal{S}}_{W_{(P,\delta)}\xi})$ smaller to resolve both issues, but keep them big enough to have enough projectives still. This can be done. We work with the category of $\mathcal{S}$-modules which are, as Fréchet spaces, direct summands of the Fréchet space $\mathcal{S}(\mathbb{Z})$ of sequences with fast decay and with $\widehat{\mathcal{S}}_{W_{(P,\delta)}\xi}$-modules which are, as Fréchet spaces, quotients of $\mathcal{S}(\mathbb{Z})$ (then we can show exactness, while continuous linear splittings are automatic in the first category, and projective modules map to projective modules in the second).

Remarkably, for the Harish-Chandra-Schwartz algebra completion $\mathfrak{S}(G)$ of the Hecke algebra $\mathcal{H}(G)$ of a reductive p-adic group G the comparison theorem is known to be true as well, by a result of Ralph Meyer. In fact one can check that all the above arguments can be made to work in this context as well.

Theorem

If π, π' be smooth tempered irreducible representations of G. If they are in distinct Harish-Chandra blocks then $\mathrm{Ext}^i_{\mathcal{H}(\mathcal{G})}(\pi, \pi') = 0$ for all i. Else let π, π' both be summands of the Harish-Chandra block $\mathrm{Ind}_P^G(\delta_t)$ for δ a discrete series character of the Levi factor L of a standard parabolic subgroup P of G, and t a unitary character of the center of L. Then $\mathrm{Ext}^i_{\mathcal{H}(\mathcal{G})}(\pi, \pi') = (\Phi_\xi(\pi)^* \otimes \Phi(\pi')^* \otimes \bigwedge^i(E^*_\xi))^{R_\xi}$ where R_ξ is the Knapp-Stein analytic R-group for the tempered induction datum $\xi = (P, \delta, t)$.

As a consequence, we can compute the Euler pairing

$$\langle \pi, \pi' \rangle^{EP}_{\mathcal{H}(G)} := \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i_{\mathcal{H}(G)}(\pi, \pi') \in \mathbb{Z}$$

between two tempered irreducible $\mathcal{H}(G)$-modules π and π':

Theorem

Let π, π' be in the same Harish-Chandra block defined by the tempered induction datum $\xi = (P, \delta, t)$ then

$$\langle \pi, \pi' \rangle^{EP}_{\mathcal{H}(G)} = \frac{1}{|R_\xi|} \sum_{r \in R_\xi} \chi_{\Phi_\xi(\pi)}(r) \chi_{\Phi_\xi(\pi')}(r^{-1}) det(1 - r)_{E_\xi}$$

$$=: \langle \Phi_\xi(\pi), \Phi_\xi(\pi') \rangle^{Ell}_{R_\xi}$$

The right hand side is called the elliptic paring of the (twisted) characters $\Phi_\xi(\pi), \Phi_\xi(\pi')$ of R_ξ.

For admissible representations π', π of G one defines:

$$\langle \pi, \pi' \rangle_G^{Ell} := \int_{\text{Ell(G)}} \theta_\pi(c^{-1})\theta_{\pi'}(c)\mathrm{d}\mu_{ell}(c)$$

where Ell(G) is the set of regular elliptic conjugacy classes of G, and θ_π, $\theta_{\pi'}$ are the distributional characters of π and π', and μ_{ell} is the Weyl integration measure on the set of regular elliptic classes.

Theorem (Arthur)

For smooth tempered irreducible characters π, π' of G one has $\langle \pi, \pi' \rangle_G^{Ell} = 0$ unless π, π' are both in the same Harish-Chandra block defined by a tempered induction datum $\xi = (P, \delta, t)$ say. In that case one has $\langle \pi, \pi' \rangle_G^{Ell} = \langle \Phi_\xi(\pi), \Phi_\xi(\pi') \rangle_{R_\xi}^{Ell}$.

Corollary (Kazhdan's orthogonality conjecture (Bezrukavnikov, Schneider-Stuhler))

For admissible characters π, π' *of G one has* $\langle \pi, \pi' \rangle_G^{Ell} = \langle \pi, \pi' \rangle_{\mathcal{H}(G)}^{EP}$.

Proof.

For smooth tempered irreducible characters this follows from Arthur's theorem and our computation of $\langle \pi, \pi' \rangle_{\mathcal{H}(G)}^{EP}$. Clearly for both pairings parabolically induced characters are in the radical. By the Langlands classification the result therefore reduces to the tempered case. □