Local injectivity for weighted Radon transforms

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Let \( m(x, \xi, \eta) \) be a smooth, positive function defined in a neighborhood of \((0, 0, 0) \in \mathbb{R}^3\). For \( f(x, y) \) supported in \( y \geq x^2 \) set

\[
R_m(\xi, \eta) = \int f(x, \xi x + \eta) m(x, \xi, \eta) dx, \quad (\xi, \eta) \approx (0, 0).
\]

**Question.** For which \( m(x, \xi, \eta) \) is it true that

\[
R_m(\xi, \eta) = 0 \quad \text{for} \quad (\xi, \eta) \approx (0, 0)
\]

implies

\[
f(x, y) = 0 \quad \text{for} \quad (x, y) \approx (0, 0).
\]

**Note.** Not true for all \( m(\xi, \eta) \).
Case $m(x, \xi, \eta) = 1$: classical Radon transform.

Global inversion formula proved by Radon in 1917:

$$f = c_n(-\Delta)^{(n-1)/2} R^* R f.$$

If $\phi(L)$ is a function defined on the mfd of hyperplanes, then

$$R^* \phi(p) = \int_{L \ni p} \phi(L) dL.$$
Helgason’s support theorem (1965) (special case):

Let $K$ be compact, convex $\subset \mathbb{R}^n$. Assume $Rf(L) = 0$ for all hyperplanes $L$ not intersecting $K$.

Then $f = 0$ outside $K$. 
Theorem. (Strichartz 1982).
Assume $f(x, y) = 0$ for $y < x^2$ and

$$Rf(\xi, \eta) = \int f(x, \xi x + \eta)dx = 0, \quad (\xi, \eta) \approx (0, 0).$$

Then $f(x, y) = 0$ for $(x, y) \approx (0, 0)$.

Proof. Set

$$G_k(\xi, \eta) = \int x^k f(x, \xi x + \eta)dx, \quad k = 0, 1, \ldots.$$

The assumption means that $G_0(\xi, \eta) = 0$. Hence

$$\partial_\xi G_0(\xi, \eta) = \int xf'_y(x, \xi x + \eta)dx = 0.$$

The last expression is equal to $\partial_\eta G_1(\xi, \eta)$, so we have also

$$\partial_\eta G_1(\xi, \eta) = \partial_\xi G_0(\xi, \eta) = 0, \quad (\xi, \eta) \approx (0, 0).$$
But $G_1(\xi, \eta) = 0$ if the line $y = \xi x + \eta$ does not meet the support of $f$, that is, if $\eta < -\xi^2/4$. Hence $G_1(\xi, \eta) = 0$ in a nbh of $(0,0)$.

This process can be continued, because the same argument shows that

$$\partial_\xi G_k(\xi, \eta) = \partial_\eta G_{k+1}(\xi, \eta) \quad \text{for all } k.$$  

So we obtain $G_k(\xi, \eta) = 0$ for all $k$ in a fixed neighborhood of the origin, which means in particular that

$$G_k(0, \eta) = \int x^k f(x, \eta) \, dx = 0 \quad \text{for all } k, \eta < \delta,$$

hence $f(x, \eta) = 0$ for all $x$ and all $\eta$ in a nbh of 0.
**Theorem.** $R_m$ is locally injective if $m(x, \xi, \eta)$ is real analytic and positive (JB and Quinto, 1987).

**Proof sketch:** If $R_m f(\xi, \eta) = 0$ in a nbh of $L(\xi, \eta)$, then

$$(z, \theta) \notin WF_A(f) \quad \text{for all} \quad (z, \theta) \in N^*(L(\xi, \eta)).$$

Here $z = (x, y)$. And this implies that

$$(z, \theta) \notin WF_A(hf) \quad \text{for all} \quad (z, \theta) \in N^*(L(\xi, \eta))$$

and every $h(x)$ that is real analytic. And this in turn implies that (choose $\xi = 0$)

$$Q_h(\eta) = \int f(x, \eta) h(x) dx \quad \text{is real analytic.}$$

But $Q_h(\eta) = 0$ if $\eta < -\xi^2$, hence $Q_h = 0$ in a nbh of $\eta = 0$. Since this is true for all real analytic $h$, it follows that $f = 0$ in a nbh of $(0, 0)$. 
The attenuated Radon transform

ECT = Emission Computed tomography:

\[ m(x, \xi, \eta) = \exp \left( - \int_{x}^{\infty} a(t, \xi t + \eta) dt \right), \]

that is,

\[ \partial_x \log m(x, \xi, \eta) = a(x, \xi x + \eta), \]

where \( a(x, y) \) is known.
Inversion formula for this class of $m(x, \xi, \eta)$ was proved by:

Arbuzov, Bukhgeim, and Kazantsev 1998

Novikov 2002

Natterer 2001

Boman and Strömberg 2004

Bal 2004

Fokas and Sung 2005
The inversion formula can be written:

$$\left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left( n(x) \left( R_{1/m}^* \Theta \tau^{-1} H \tau R_m + R_{1/m}^* \Theta \bar{\tau}^{-1} H \bar{\tau} R_m \right) f(x) \right) = 8\pi n(x) f(x).$$

Here $H$ denotes the Hilbert transform on the real line, $\Theta$ denotes the operator of multiplication by $\theta_1 + i\theta_2$, and $\tau(\theta, p)$ and $n(x)$ are a complex valued function that are computed from $m$. 
Proof idea:

Use the familiar notation \( x = (x_1, x_2), \theta = (\theta_1, \theta_2) \in S^1 \) and

\[
Rf(\theta, p) = \int_{x \cdot \theta = p} f(x)m(x, \theta)ds, \quad (\theta, p) \in S^1 \times \mathbb{R}.
\]

Step 1. An inversion formula is proved for complex valued \( m(x, \theta) \) for which \( \theta \mapsto m(x, \theta) \) is the boundary value of an analytic function in the unit disk for every \( x \).

Step 2. Prove that every \( m(x, \theta) \) of attenuation type can be written

\[
m(x, \theta) = b(\theta, p)a(x)m_0(x, \theta), \quad p = x \cdot \theta,
\]

where \( m_0(x, \theta) \) satisfies the condition in Step 1.
Recall that
\[ R_m(\xi, \eta) = \int f(x, \xi x + \eta)m(x, \xi, \eta)dx. \]

**Theorem.** (Gindikin 2009) Assume that \( m(x, \xi, \eta) \) satisfies
\[ m'_\xi(x, \xi, \eta) - x m'_\eta(x, \xi, \eta) = (x a(\xi, \eta) + b(\xi, \eta))m(x, \xi, \eta) \]
for some functions \( a(\xi, \eta) \) and \( b(\xi, \eta) \). Then
\[ cf(0, 0) = \int \int \frac{(\partial_\eta + a(\xi, \eta))R_m(\xi, \eta)}{m(0, \xi, \eta)} \frac{d\eta}{\eta} d\xi. \]
The operator $\partial_\xi - x\partial_\eta$ appearing in Gindikin’s condition is the directional derivative in the direction of rotating the line through a fixed point $(x, y)$. Because

$$\partial_\xi(m(x, \xi, y - \xi x)) = m'_\xi(x, \xi, y - \xi x) - x m'_\eta(x, \xi, y - \xi x).$$
Let us compute an expression for the adjoint $R_m^*$ of $R_m$.

$$\langle R_m f, \phi \rangle = \int \int R_m f(\xi, \eta) \phi(\xi, \eta) d\xi d\eta = \langle f, R_m^* \phi \rangle,$$

where

$$R_m^* \phi(x, y) = \int \phi(\xi, -\xi x + y) m(x, \xi, -\xi x + y) d\xi$$
Theorem. (JB 2009) Assume \( m(x, \xi, \eta) \) satisfies Gindikin’s condition, that is,

\[
m'_\xi(x, \xi, \eta) - x m'_\eta(x, \xi, \eta) = (x a(\xi, \eta) + b(\xi, \eta))m(x, \xi, \eta).
\]

Then local injectivity holds for \( R_m \).

Proof. Set

\[
G_k(\xi, \eta) = \int x^k f(x, \xi x + \eta)m(x, \xi, \eta)dx.
\]

Using induction I shall prove that

\[
G_k(\xi, \eta) = 0, \quad (\xi, \eta) \approx (0, 0), \quad k = 0, 1 \ldots.
\]

To do this I shall prove

\[
(\partial_\xi - b(\xi, \eta))G_k = (\partial_\eta - a(\xi, \eta))G_{k+1}.
\]
In fact
\[ \partial_\xi G_k = \int x^{k+1} f'_y m \, dx + \int x^k f m'_\xi \, dx, \quad \text{and} \]
\[ \partial_\eta G_{k+1} = \int x^{k+1} f'_y m \, dx + \int x^k f m'_\eta \, dx. \]
Hence
\[
\partial_\xi G_k - \partial_\eta G_{k+1} \\
= \int x^k f(\ldots)(m'_\xi - x m'_\eta) \, dx \\
= \int x^k f(\ldots)(x a(\xi, \eta) + b(\xi, \eta)) m \, dx \\
= a G_{k+1} + b G_k.
\]
Assume \( G_k(\xi, \eta) = 0 \). It follows that
\[ \partial_\eta G_{k+1}(\xi, \eta) + a(\xi, \eta) G_{k+1}(\xi, \eta) = 0. \]
But \( G_{k+1}(\xi, \eta) = 0 \) if \( \eta < -\xi^2/4 \). Hence \( G_{k+1}(\xi, \eta) = 0 \) in a nbh of the origin.
**Theorem.** (JB 1993) There exists a compactly supported function $f$ in the plane, not identically zero, and a positive, smooth weight function $m_L(x, y)$ such that

$$
\int_L fm_L ds = 0 \quad \text{for all lines } L \text{ in the plane.}
$$
**Theorem.** There exists $m(x, \xi, \eta) \in C^\infty$, $> 0$, and $f \in C^\infty$ such that $\text{supp } f \subset \{|x| \leq y\}$, $(0,0) \in \text{supp } f$, and

$$
\int f(x, \xi x + \eta)m(x, \xi, \eta)dx = 0 \quad \text{for } |\xi| < 1/2, \eta < 1.
$$

**Proof.** Choose $f(x, y)$ first, then define $m(x, \xi, \eta)$ by

$$
m(x,y,L) = 1 - c(L)f(x,y), \quad (x,y) \in L.
$$

Then

$$
R_m f(L) = \int_L f(1 - c_L f)dx = \int_L f dx - c(L) \int_L f^2 dx,
$$

so $R_m f = 0$ iff

$$
c(L) = \frac{\int_L f dx}{\int_L f^2 dx}.
$$
Choose
\[ f_k(x, y) = \phi(2^k x, 2^k y) \cos 4^k x, \]
where \( \phi \in C^\infty \), is supported in the rectangle
\[ |x| \leq 1, \quad 1 \leq y \leq 3, \]
\( \phi = 1 \) in a slightly smaller rectangle, and
\[ f(x, y) = \sum_{k=1}^{\infty} f_k(x, y)/k!. \]
The following is obvious:
\[ f \in C^\infty \]
\[ \text{supp } f \subset \{|x| \leq y\} \]
\[ \int_{L(\xi, \eta)} f^2 \, dx > 0 \quad \text{if } \eta > 0. \]
Hence

\[ c(L) = \frac{\int_L f \, dx}{\int_L f^2 \, dx} \in C^\infty, \quad \eta > 0. \]
Lemma. For every $p$ there exists $C_p$ such that

$$\left| \partial^\alpha_{(\xi,\eta)} \int_{L(\xi,\eta)} f_k dx \right| \leq C_p 2^{-kp}, \quad |\alpha| \leq p.$$ 

Note: if $L$ intersects $\text{supp} f_k$ and $|\xi| < 1/2$, then $\eta \sim 2^{-k}$.

Proof.

$$\int_{L(\xi,\eta)} f_k dx = \int \phi(2^k x, 2^k (\xi x + \eta)) \cos 4^k x dx.$$ 

Partial integration $p$ times gives (if $p$ is even)

$$\int_{L(\xi,\eta)} f_k dx = \pm 4^{-kp} \int (\cos 4^k x) \partial_x^p \left( \phi(2^k x, 2^k (\xi x + \eta)) \right) dx.$$ 

$$\left| \int_{L(\xi,\eta)} f_k dx \right| \leq 4^{-kp} C_p 2^{kp} = C_p 2^{-kp}.$$
Next use the fact that
\[ \int_{L(\xi,\eta)} f^2 dx \geq c2^{-k}/(k!)^2, \quad \eta \sim 2^{-k}. \]

Recall that \(2^{-k} \sim \eta\) if \(L(\xi, \eta)\) meets the support of \(f\). Write down obvious estimates for derivatives of \(\int_{L(\xi,\eta)} f^2 dx\). It follows that each derivative
\[ \partial^\alpha_{(\xi,\eta)} c(L(\xi,\eta)) \to 0 \quad \text{as} \quad \eta \to +0, \]
hence \(c(L)\) is \(C^\infty\) and so is
\[ m(x, y, L) = c(L)f(x, y). \]
The weight function \( m(x, \xi, \eta) = 1 - c(L)f(x, y) \) is close to 1 in a neighborhood of \((0, 0, 0)\). In fact we can make it as close as we wish to 1 by introducing a parameter \( \lambda \) in the definition of \( f_k \),

\[
f_k(x, y) = \phi(2^k x, 2^k y) \cos 4^k \lambda x,
\]

and making \( \lambda \) sufficiently large.

More interesting, starting with an arbitrary \( m_0(x, y, L) \) and choosing

\[
m(x, y, L) = m_0(x, y, L) - c(L)f(x, y),
\]

where

\[
c(L) = \int_L fm_0 dx \bigg/ \int f^2 dx
\]

we can make \( m(x, y, L) \) as close as we wish to an arbitrary given \( m_0(x, y, L) \). Thus
Theorem. (JB 2010) The set of $m(x, \xi, \eta)$ for which $R_m$ is not locally injective is dense in the set of smooth, positive weight functions.

On the other hand, it is well known that the set of weight functions $m$ for which $R_m$ is globally injective is open in the $C^1$ topology.
Given a 2-parametric curve family

\[ y = u(x, \xi, \eta) \approx \eta + \xi x + \mathcal{O}(x^2) \]

and a weight function \( m(x, \xi, \eta) \), set

(1) \[ Rf(\xi, \eta) = \int f(x, u(x, \xi, \eta))m(x, \xi, \eta)dx, \]

and ask the same question as before.

To define \( R \) invariantly (Gelfand, Helgason, Guillemin), denote \((x, y)\)-space by \( X \) and the space of curves by \( \Gamma \). Then the equation \( y = u(x, \xi, \eta) \) defines a hypersurface \( Z \) in the product space \( X \times \Gamma \). Invariantly, the hypersurface \( Z \) is the incidence relation:

\[ \{(p, \gamma); p \in \gamma\}. \]
Using the projections

\[ \pi_X : (p, \gamma) \mapsto p, \quad \pi_\Gamma : (p, \gamma) \mapsto \gamma, \]

we can define two kinds of fibers on \( Z \):

\[ \pi_\Gamma^{-1}(\gamma) \quad \text{and} \quad \pi_X^{-1}(p). \]

and projecting those down to \( X \) and \( \Gamma \) we obtain

\[ \pi_X(\pi_\Gamma^{-1}(\gamma)) \quad \text{and} \quad \pi_\Gamma(\pi_X^{-1}(p)). \]
If $\mu$ is a measure on $Z$ and $f \in C(X)$, $\varphi \in C_c(\Gamma)$ we can form
\[
\langle \mu, (f \circ \pi_X)(\varphi \circ \pi_\Gamma) \rangle = \langle Rf, \varphi \rangle = \langle f, R^*\varphi \rangle.
\]
If $\mu = m(x, \xi, \eta)\, dx\, d\xi\, d\eta$ we get back the expression (1) for $Rf$.

Introduce the function
\[
q(x, \xi, \eta) = \frac{u_\xi'(x, \xi, \eta)}{u_\eta'(x, \xi, \eta)} = \frac{x + \ldots}{1 + \ldots} = x + \ldots.
\]
Consider the following condition on $m(x, \xi, \eta)$:

$$\partial_\eta(qm) - \partial_\xi m = (a_1 q + b_1)m,$$

where $a_1$ and $b_1$ are functions that depend only on $(\xi, \eta)$.

To formulate the condition on $Z$ we define $Z$ by an equation $\eta = \rho(\xi, x, y)$. The condition on $Z$ is as follows: the functions $\eta = \rho(\xi, x, y)$ are solutions of an ODE

$$\eta'' = \Psi(\xi, \eta, \eta'),$$

where $\Psi(\xi, \eta, p)$ is a polynomial in $p$ of degree at most 3.

**Theorem.** $R$ is locally injective if (2) and (3) are satisfied.
The condition on $\mu = m(x, \xi, \eta)dx d\xi d\eta$ can be invariantly expressed as follows:

there exists a 1-form $\sigma$ on $\Gamma$ such that

$$V(\mu) = (\pi^*_\Gamma(\sigma) \lrcorner V)\mu$$

for all vector fields $V$ that are tangent to the fibers $\pi_{X}^{-1}(p)$.

The operation of a vector field on a density is defined by $\langle V(\mu), \varphi \rangle = -\langle \mu, V(\varphi) \rangle$. 
A 2-parametric family of curves in the plane can be defined by a second order differential equation

\[ y'' = \Phi(x, y, p), \quad p = \frac{dy}{dx}. \]

**Proposition.** The class of differential equations (4) for which \( p \mapsto \Phi(x, y, p) \) is a polynomial of degree \( \leq 3 \) is invariant under smooth coordinate transformations in the plane.

See Arnold, Geometric aspects of the theory of ODE, chapter 1, §6.