Measures of relative complexity

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Talking about problems

Many problems ask:

- for a YES or NO answer
- to determine a value y

given an input value x.

Data and the solution to many problems can often be coded into finite or infinite binary sequences.

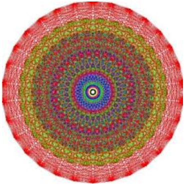
We work in the Cantor space.

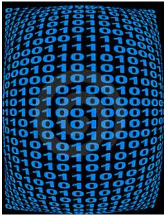
Examples of problems

Presentations of groups

Diophantine equations

Satisfiability of propositional formulas





Verification of programs

Optimization

Raw data emanated from a physical source

Reducibilities and hierarchies

Reducibilities are preorders $\leq$. They

- induce an equivalence relation denoting identical complexity
- reorder the continuum in terms of complexity
- are one of the main tools for measuring complexity

Hierarchies: sequences of classes of increasing complexity

Arithmetical hierarchy

Polynomial hierarchy

Jump hierarchy

Difference hierarchy

Complexity of streams

An infinite binary sequence may be:

Theoretically computable

Definable (e.g. in arithmetic)

Feasibly computable

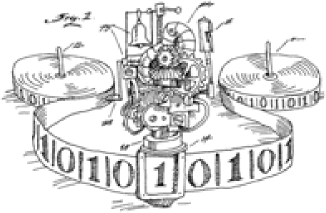
Incomputable (but possibly predictable)

Random (with no algorithmic structure)

Quantifying intractability

Some problems are intractable.

- *A* is computable in *B*
- *A* is definable in *B*
- *A* is feasibly computable in *B*
- *A* is random in *B*
- *A* less random than *B*



Complexity definitions relativize to a parameter *X*.

A stream that is used as external information utilized in the course of a computation is called an oracle.

Relative computability and degrees

$A \leq_T B$ denotes A is computable from B

$A \equiv_T B$ denotes $A \leq_T B$ and $B \leq_T A$

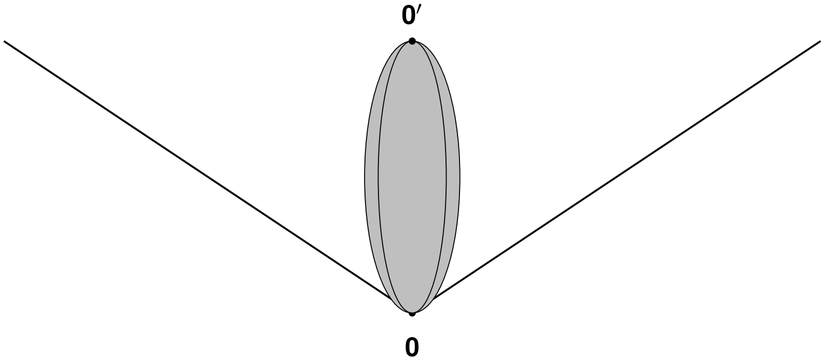
The equivalence classes are known as the Turing degrees.

They form a partially ordered set.

Two streams in the same degree are equally tractable.

A degree formalizes the concept of information content.

Degrees of unsolvability



Other classical reducibilities and degrees

- Strong reducibilities: truth-table, many-one, Lipschitz,...
- Enumeration reducibility
- Arithmetical reducibility

A Guiding Principle:

Computability is some kind of definability and vice-versa.

People in degree theory

50s

60s,70s

70s, 80s

Themes in degree theory

Algebraic study: embeddings, ideals, automorphisms

Logical study: decidability and complexity of the theory of degrees, definability.

Global versus local structure and theory

Computably enumerable sets and degrees

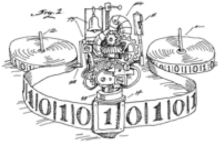
Category and Measure (genericity, randomness)

Randomness and Incompressibility

A stream is random if all of its initial segments are incompressible.

Descriptions should be given in an algorithmic way:

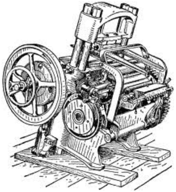
If M is a Turing machine and $M(\sigma) = \tau$, then σ is an M-description of τ.



The complexity of a binary string is the length of its shortest description.

Kolmogorov complexity of strings

Let $|M|$ be the **size** of the machine M.



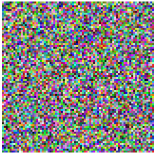
Let $K_M(\sigma)$ be the complexity of σ w.r.t. M.

The complexity of σ is the least sum $|M| + K_M(\sigma)$ where M ranges over all machines.

Let $K(\sigma)$ denote the complexity of σ.

A string is c-compressible if it has a description that is shorter than its length by at least c bits.

Algorithmic randomness



A stream X is random if there is a constant c such that $K(X \upharpoonright_n) \geq n - c$ for all n.

This notion of randomness is robust:

- Coincides with other approaches (betting strategies, statistics)
- Random reals form a set of measure 1
- Meets laws of large numbers, normality etc.
- Relativizes giving randomness of various strengths

On the other end: trivial initial segment complexity

The lowest possible initial segment complexity: $K(0^n) \approx K(n)$.

An interesting fact:

There are noncomputable streams with trivial initial segment complexity.

These streams are as simple as 000000... but incomputable!

Measures of relative randomness

Segment-by-segment complexity comparison:

> *Y is at least as random as X if its initial segments have higher complexity than those of X.*

…if $\forall n\ K(X \upharpoonright_n) < K(Y \upharpoonright_n) + c$ for some c.

Notation: $X \leq_K Y$

- $\leq_K$ is a weak reducibility
- Induced degree structure: the K-degrees

Differences and similarities of $\leq_K$ with $\leq_T$

- Same complexity as Turing reducibility
- Countable and uncountable degrees
- Lack of underlying reduction...

  (no algorithmic procedure connecting the two arguments)
- Countable and uncountable lower cones
- Structural differences
- Lack of a join operator...even upper bounds!

Measuring compressing power

A string may be X-incompressible but Y-compressible.

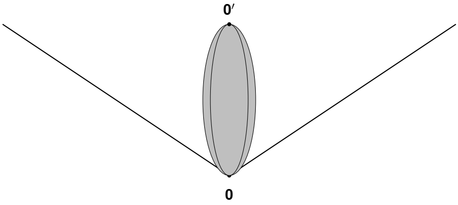
Y can compress at least as well as X if it compresses every string more than X modulo a fixed number of bits.

... if $\exists c \forall \sigma \; K^Y(\sigma) \leq K^X(\sigma) + c$

... denoted by $X \leq_{LK} Y$.

$\leq_T$ implies $\leq_{LK}$ (but not vice-versa).

Degrees of compressibility: *LK*



- Same complexity as $\leq_T$ (3rd level of arithmetical complexity)
- Countable degrees
- Countable and uncountable lower cones
- Lack of a natural supremum operator
- Structural differences with $\leq_T$
- Large chains: a perfect set forming a $\leq_{LK}$-chain!

Surprising coincidence: $\mathbf{0}_{LK} = \mathbf{0}_K$

zero compressing power $=$ *trivial initial segment complexity*

...one of the main results in the last ten years in this area.

A stream cannot compress strings more than a computable oracle if and only if its initial segments can be described as easily as 0000...

Reducibilities $\leq_K$ and $\leq_{LK}$ on random strings

For X, Y random: $X \leq_K Y \iff Y \leq_{LK} X$

Informally, in the world of random streams...

One stream is more random than another if and only if it has less compressing power.

In particular...

The more random a stream is, the less oracle power it has.

...coded information introduces structure in a binary stream.

Partial relativization and reducibilities

Some reducibilities come from partial relativization of a lowness notion.

Example: X is low for K if it has zero compressing power.

Faithful relativization:

'X is low for random relative to Y': $X \oplus Y \leq_{LK} Y$ ... not transitive!

Partial relativization: $X \leq_{LK} Y$ ... useful!

Further examples: Jump traceability, different randomness notions,...

Current research

Reducibilities associated with variations of randomness:

- Schnorr randomness
- Higher randomness: Borel and beyond

Measuring randomness of effective reals: Solovay reducibility

(For effective reals)

Hardness of approximation equals degree of randomness

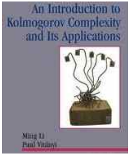
Various basic questions remain open.

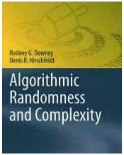
Is there a maximal K-degree?

Research in local substructures is also an interesting direction.

Computably enumerable sets: Density? Upper bounds ?

Further reading





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