Wall-crossing, dilogarithm identities, and the QK/HK correspondence

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Based on [Alexandrov, D.P. , Pioline, 1110.0466]

Motivation

Instanton corrections to the hypermultiplet moduli space in $\mathcal{N} = 2$ Calabi-Yau string vacua.

[Becker, Becker, Strominger]

→ **D-instantons**

→ ~~**NS5-instantons**~~

Key problem is wall-crossing

In this talk we suppress NS5-effects

Two main players:

Hypermultiplet moduli space for type II on X	Coulomb branch of SW-theory on S^1
$\mathcal{M}$	$\mathcal{M}'$
quaternion-Kähler	hyperkähler
D-instanton corrections	BPS-instanton corrections
(ED-branes wrapping $\mathcal{C} \subset X$)	(Euclidean BPS-states wrapping S^1)
Wall-crossing: complex contact geometry	Wall-crossing: complex symplectic geometry

How to "lift" the Kontsevich-Soibelman WCF from symplectic geometry to contact geometry?

How to "lift" the Kontsevich-Soibelman WCF from symplectic geometry to contact geometry?

Use the QK/HK correspondence:

$$\begin{tikzcd} \mathcal{M} \arrow[r, leftrightarrow] & \mathcal{L} \arrow[d] \\ & \mathcal{M}' \end{tikzcd}$$

Stringy moduli space (QK)

Hyperholomorphic line bundle over the Coulomb branch (HK)

Wall-crossing in string theory described in terms of $\mathcal{L}$!

Outline

- The hypermultiplet moduli space in II/CY3
- The QK/HK correspondence
- Wall-crossing in Seiberg-Witten theories
- Wall-crossing in N=2 string vacua
- Conclusions and Discussion

Compactification of type II on a Calabi-Yau threefold $\mathcal{X}$

$\mathcal{N} = 2, D = 4$ **supergravity with moduli space**

$$\mathcal{M}_{\mathrm{SK}} \times \mathcal{M}$$

[Bagger, Witten]

Special Kähler manifold

Quaternion-Kähler manifold

$\mathrm{Hol} \subset SU(2) \times Sp(n)$

$\dim_{\mathbb{R}} = 4n$

Useful to exploit T-duality: $\mathcal{M}$ **is the image of the "c-map" from** $\mathcal{M}_{\rm SK}$

[Ferrara, Sabharwal][Cecotti, Ferrara, Girardello]

	IIA/X	IIB/X
$D=4$		$\mathcal{M}_{\rm SK}$
$D=3$		

VM moduli space
(complex structure of $\mathcal{X}$)

Useful to exploit T-duality: $\mathcal{M}$ **is the image of the "c-map" from** $\mathcal{M}_{\rm SK}$

[Ferrara, Sabharwal][Cecotti, Ferrara, Girardello]

$$
\begin{tikzcd}
 & \mathrm{IIA}/X & & \mathrm{IIB}/X \\
D=4 & \mathcal{M} \arrow[d, "S^1"'] & & \mathcal{M}_{\rm SK} \arrow[d, "S^1"] \\
D=3 & \mathcal{M} \arrow[rr, leftrightarrow, "\text{T-duality}", "\text{“c-map”}"'] & & \widetilde{\mathcal{M}}_{\rm SK}
\end{tikzcd}
$$

→ **map from special Kähler to quaternion-Kähler**

→ **maps black holes to instantons** [Polyakov][Seiberg,Witten][Gunaydin,Neitzke,Pioline,Waldron]

$$\mathrm{D3}/\mathcal{C}_3 \longrightarrow \mathrm{ED2}/\mathcal{C}_3$$

In D=3 we have a unified description of type II/CY

"Baby version" of the c-map in $\mathcal{N} = 2$ Seiberg-Witten theories

SW-theory on the Coulomb branch

$$\begin{array}{ll} D = 4 & \mathcal{B} \\ & \Big\downarrow S^1 \\ D = 3 & \mathcal{M}' \end{array}$$

→ map from (rigid) special Kähler $\mathcal{B}$ to hyperkähler $\mathcal{M}'$ (rigid c-map)

→ maps Euclidean BPS-states to instantons [Seiberg,Witten]

In $\mathcal{N} = 2, D = 4$ field theories, compactification on S^1 holds the key to understand wall-crossing of the BPS-spectrum. [Gaiotto, Moore, Neitzke]

The QK/HK correspondence

To any QK-manifold $\mathcal{M}$ of real dimension 4n, with a rotational isometry, one can associate a dual HK-manifold $\mathcal{M}'$ of the same real dimension, with a rotational isometry and equipped with a hyperholomorphic line bundle $\mathscr{L}$ and connection λ .

The original idea is due to unpublished work by Neitzke & Pioline in 2008.

This correspondence was also discovered independently by Haydys in 2007.

Some examples also implicit in the work of Rocek, Vafa, Vandoren in 2006.

It is currently being investigated by Hitchin as well as by Swann & Macia.
(see lectures by Hitchin in Oxford last week, and talk at INI in July, 2011)

Some clarifications and new examples recently found by Vandoren.

The QK/HK correspondence

To any QK-manifold $\mathcal{M}$ of real dimension 4n, with a rotational isometry, one can associate a dual HK-manifold $\mathcal{M}'$ of the same real dimension, with a rotational isometry and equipped with a hyperholomorphic line bundle $\mathscr{L}$ and connection λ .

→ **Hyperholomorphic: $\mathscr{L}$ is holomorphic with respect to a whole $\mathbb{C}P^1$ worth of complex structures.**

$c_1(\mathscr{L})$ **is of type** $(1,1)$ **in all complex structures.**

→ **Generalizes the notion of a Yang-Mills instanton to hyperkähler spaces.**

[Capria,Salamon][Gocho, Nakajima][Verbitsky]

The QK/HK correspondence

Rough idea of the construction:

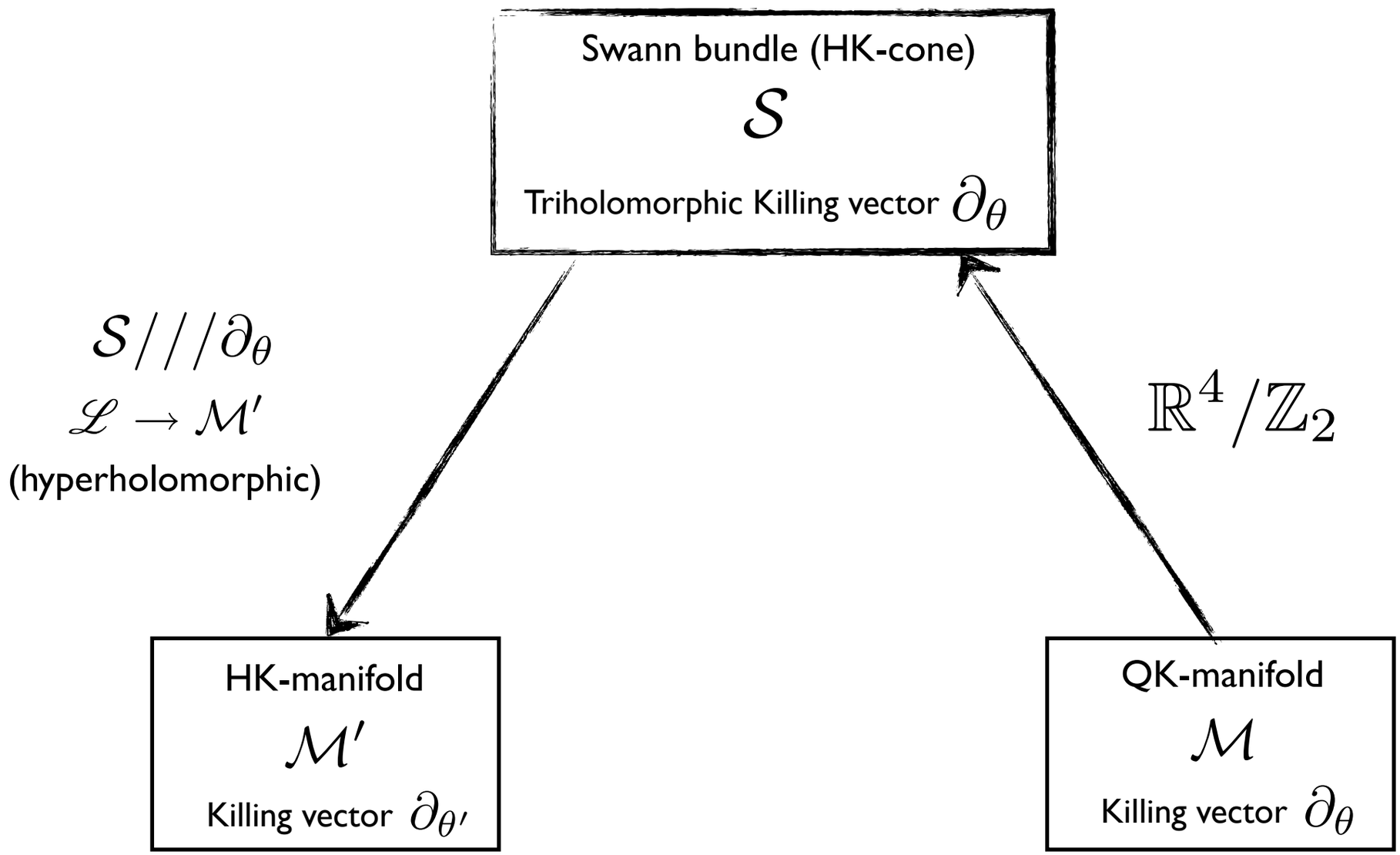
(i) lift the quaternionic isometry ∂_θ on $\mathcal{M}$ to a triholomorphic action on the Swann bundle (or hyperkähler cone):

$$\mathbb{R}^4/\mathbb{Z}_2 \to \mathcal{S} \to \mathcal{M}$$

(ii) perform a hyperkähler quotient with respect to ∂_θ

(iii) the result is a hyperkähler manifold $\mathcal{M}'$ with an isometric circle action which rotates the complex structures, and carries a connection λ such that $\mathscr{F} = \mathrm{d}\lambda$ is $(1,1)$ in all cplx. structures.

The QK/HK correspondence



Some examples

Ex. 1. $\mathcal{M}' = \mathbb{R}^4 \longleftrightarrow \mathcal{M} = SU(2,1)/(SU(2) \times U(1))$

Universal hypermultiplet moduli space

Ex. 2. **More generally, if $\mathcal{M}'$ is obtained from the rigid c-map, then the dual space $\mathcal{M}$ is obtained from the local (1-loop corrected) c-map.**

The QK/HK correspondence in twistor space

- The HK-twistor space

→ **Hitchin's construction implies that the HK-metric on $\mathcal{M}'$ can be encoded in the complex symplectic structure on the twistor space**

$$\mathcal{Z}' = \mathcal{M}' \times \mathbb{C}P^1_\zeta$$

$$\omega' = \omega'_+ - i\zeta\omega'_3 + \zeta^2\omega'_-$$

→ **Locally, in $U_i \subset \mathcal{Z}'$, there exists cplx Darboux coords $(\eta^\Lambda_{[i]}, \mu^{[i]}_\Lambda)$ such that:**

$$\omega' = \mathrm{d}\eta^\Lambda_{[i]} \wedge \mathrm{d}\mu^{[i]}_\Lambda$$

→ **On overlapping patches (U_i, U_j), the Darboux coords are related by cplx. symplectomorphisms, determined by a generating function $H'(\eta, \mu, \zeta)$**

→ **U(1) isometry implies** $H'(\eta, \mu, \zeta) \equiv H'(\eta, \mu)$

The QK/HK correspondence in twistor space

- The QK-twistor space

→ The QK-metric on $\mathcal{M}$ can be encoded in the complex contact structure on the twistor space [Salamon]

$$\mathbb{C}P^1_t \to \mathcal{Z} \to \mathcal{M}$$

→ Locally, there exists cplx. Darboux coords $(\xi^\Lambda_{[i]}, \tilde{\xi}^{[i]}_\Lambda, \alpha^{[i]})$ such that the contact 1-form is given by:

$$\mathcal{X}^{[i]} = \mathrm{d}\alpha^{[i]} + \tilde{\xi}^{[i]}_\Lambda \mathrm{d}\xi^\Lambda_{[i]} - \xi^\Lambda_{[i]} \mathrm{d}\tilde{\xi}^{[i]}_\Lambda$$

[Neitzke, Pioline, Vandoren][Alexandrov, Saueressig, Pioline, Vandoren]

→ On overlapping patches (U_i, U_j), the Darboux coords are related by cplx. contactomorphisms, determined by a generating function $H(\xi, \tilde{\xi}, \alpha)$

→ U(1) isometry implies $H(\xi, \tilde{\xi}, \alpha) \equiv H(\xi, \tilde{\xi})$

[Alexandrov, Saueressig, Pioline, Vandoren]

HK vs. QK

twistor space $\mathcal{Z}' \to \mathcal{M}'$	twistor space $\mathcal{Z} \to \mathcal{M}$
$\mathcal{Z}' = \mathcal{M}' \times \mathbb{C}P^1_\zeta$	$\mathbb{C}P^1_t \to \mathcal{Z} \to \mathcal{M}$
complex symplectic	**complex contact**
$(\eta^\Lambda(\zeta), \mu_\Lambda(\zeta), \zeta)$	$(\xi^\Lambda(t), \tilde{\xi}_\Lambda(t), \alpha(t))$
$\omega' = \mathrm{d}\eta^\Lambda \wedge \mathrm{d}\mu_\Lambda$	$\mathcal{X} = \mathrm{d}\alpha + \tilde{\xi}_\Lambda \mathrm{d}\xi^\Lambda - \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda$
$\mathcal{M}'_\zeta \cong (\mathbb{C}^\times)^{2n}$	$\mathcal{M}_t$ **not a complex manifold**

The QK/HK correspondence in twistor space

- The correspondence

For a dual pair of manifolds $(\mathcal{M}, \mathcal{M}')$, the twistor spaces are related via their Darboux coordinates according to:

$$\xi^{\Lambda}_{[i]}(t) = \eta^{\Lambda}_{[i]}(\zeta)$$

$$\tilde{\xi}^{[i]}_{\Lambda}(t) = \mu^{[i]}_{\Lambda}(\zeta)$$

$$t = \zeta e^{-i\theta'}$$

The QK/HK correspondence in twistor space

- The correspondence

For a dual pair of manifolds $(\mathcal{M}, \mathcal{M}')$, the twistor spaces are related via their Darboux coordinates according to:

$$\xi^{\Lambda}_{[i]}(t) = \eta^{\Lambda}_{[i]}(\zeta) \qquad \tilde{\xi}^{[i]}_{\Lambda}(t) = \mu^{[i]}_{\Lambda}(\zeta) \qquad t = \zeta e^{-i\theta'}$$

The twistor spaces $(\mathcal{Z}, \mathcal{Z}')$ are described by the same transition functions:

$$H(\xi, \tilde{\xi}) \equiv H'(\eta, \mu)$$

Note: This only works if $(\mathcal{M}, \mathcal{M}')$ have U(1) isometries!

The QK/HK correspondence in twistor space

- The correspondence

But what is the meaning of the contact coordinate α on the HK-side?

→ **It corresponds to a holomorphic section**

$$\Upsilon(\zeta) = e^{\pi i \alpha(\zeta)}$$

of a $\mathbb{C}^\times$ -bundle $\mathscr{L}_{\mathcal{Z}'} \to \mathcal{Z}'$.

The bundle $\mathscr{L}_{\mathcal{Z}'}$ has the crucial property of being trivial along the $\mathbb{C}P^1_\zeta$ -fiber.

By a standard construction in twistor theory, $\mathscr{L}_{\mathcal{Z}'}$ then descends to a hyperholomorphic bundle $\mathscr{L} \to \mathcal{M}'$ with connection

$$\lambda = \bar{\partial}^{(\zeta)}\alpha + \partial^{(\zeta)}\bar{\alpha}$$

The QK/HK correspondence in twistor space

- The correspondence

By a standard construction in twistor theory, $\mathscr{L}_{\mathcal{Z}'}$ then descends to a hyperholomorphic bundle $\mathscr{L} \to \mathcal{M}'$ with connection

$$\lambda = \bar{\partial}^{(\zeta)}\alpha + \partial^{(\zeta)}\bar{\alpha}$$

Curvature of $\mathscr{L}$ is manifestly of type $(1,1)$ in all cplx. structures:

$$\mathscr{F} = \mathrm{d}\lambda = 2\pi i \partial\bar{\partial}^{(\zeta)}\,\Im\alpha \qquad\qquad c_1(\mathscr{L}) = \left[\frac{\mathscr{F}}{2\pi}\right]$$

Wall-crossing in Seiberg-Witten theories

We consider N=2 theories with rank r gauge group G , broken to $U(1)^r$ on the Coulomb branch. (no flavors)

Moduli space of vacua: $\mathcal{B} \ni z^i, i = 1, \ldots, r$ **(rigid special Kähler)**

Local system of e-m charge lattices: $\Gamma \to \mathcal{B}$ $\qquad \Gamma_z \cong \mathbb{Z}^{2r}$

Charges: $\gamma = (p^\Lambda, q_\Lambda) \in \Gamma_z \qquad (\Lambda = 0, 1, \ldots, r-1)$

Central charge function: $Z' : \Gamma \to \mathbb{C}$

$$Z'_\gamma = q_\Lambda Z^\Lambda(z) - p^\Lambda F_\Lambda(z)$$

Wall-crossing in Seiberg-Witten theories

Compactification on S^1 yields a sigma model in D=3 with hyperkähler target space $\mathcal{M}'$.

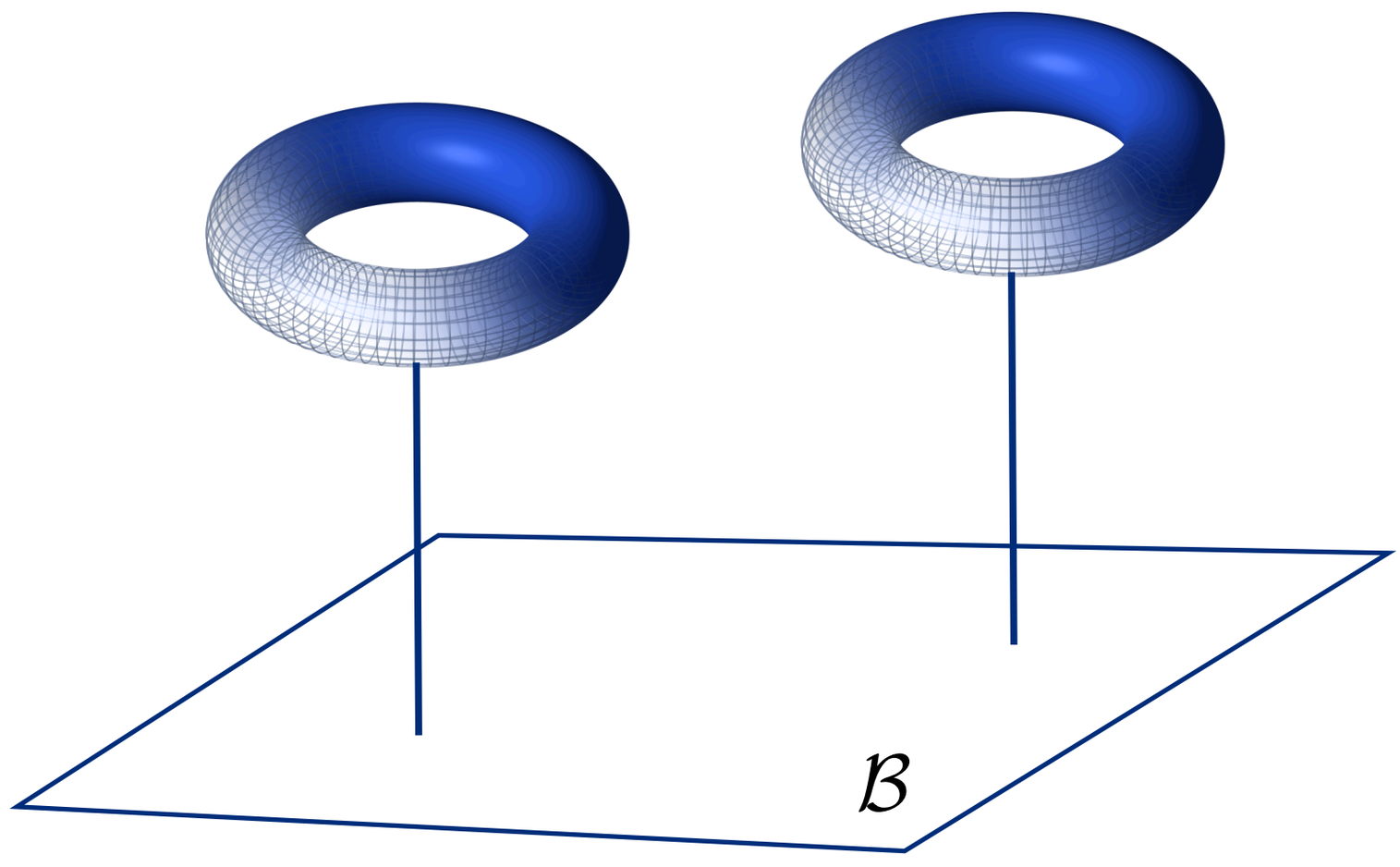
[Seiberg, Witten]

The new moduli space $\mathcal{M}'$ is a (twisted) torus fibration over $\mathcal{B}$.

Torus fibers $\mathcal{T}_z = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$ parametrized by the Wilson lines $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$

$$\zeta^\Lambda = \oint_{S^1} \mathcal{A}_4^\Lambda dx^4$$

$$\tilde{\zeta}_\Lambda = \oint_{S^1} \tilde{\mathcal{A}}_\Lambda^4 dx^4$$



In the large radius limit, the metric on $\mathcal{M}'$ takes the semi-flat form:

$$g^{\mathrm{sf}} = R\, g_{\mathcal{B}} + \frac{1}{R}\, g_{\mathcal{T}}$$

$\longrightarrow$ **Image of the rigid c-map from $\mathcal{B}$.**

At finite R there will be instanton corrections to the semiflat metric, arising from Euclidean BPS-states wrapping the circle.

One-instanton approximation:

$$g \sim g^{\mathrm{sf}} + \Omega(\gamma; z) e^{-R|Z_\gamma(z)| + 2\pi i (q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)}$$

BPS-index **central charge**

BPS-index: $\Omega : \Gamma \to \mathbb{Z}$

$$\Omega(\gamma; z) = -\frac{1}{2}\mathrm{Tr}(2J_3)^2(-1)^{2J_3}$$

(second helicity supertrace)

Locally continuous function on $\mathcal{B}$ but jumps discontinuously on real co-dimension one subspaces where the phases of the central charges align: [Cecotti,Fendley, Intriligator,Vafa][Seiberg,Witten][Denef, Moore]

$$W(\gamma_1, \gamma_2) = \{z \in \mathcal{B} \,|\, Z_{\gamma_1}(z)/Z_{\gamma_2}(z) \in \mathbb{R}_+\}$$

Decay: $\gamma \to \gamma_1 + \gamma_2$

Wall-crossing formula: determine $\Omega(\gamma; z_+) - \Omega(\gamma; z_-)$

The Kontsevich-Soibelman wall-crossing formula

Introduce a complex symplectomorphism:

$$U_\gamma(z) : \mathcal{X}'_{\gamma'} \to \mathcal{X}'_{\gamma'} \left(1 - \mathcal{X}'_\gamma\right)^{\Omega(\gamma;z)\langle\gamma,\gamma'\rangle}$$

$\mathcal{X}_\gamma$ **are holomorphic Fourier modes on the complex torus** $\mathcal{T}_{\mathbb{C}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{C}^\times$

Gaiotto-Moore-Neitzke: identify $\mathcal{T}_{\mathbb{C}}$ **with the fiber** $\mathcal{M}'_\zeta$ **of** $\mathcal{Z}' \to \mathbb{C}P^1_\zeta$

$$\mathcal{X}_\gamma = e^{-2\pi i\left(q_\Lambda \eta^\Lambda(\zeta) - p^\Lambda \mu_\Lambda(\zeta)\right)}$$

Recall that $(\eta^\Lambda(\zeta), \mu_\Lambda(\zeta))$ **are complex Darboux coords on** $\mathcal{Z}'$

The Kontsevich-Soibelman wall-crossing formula

The formula:

$$\prod_{\substack{m_1 \geq 0, m_2 \geq 0, \\ m_1/m_2 \downarrow}} U_{m_1\gamma_1 + m_2\gamma_2}(z_+) = \prod_{\substack{m_1 \geq 0, m_2 \geq 0, \\ m_1/m_2 \uparrow}} U_{m_1\gamma_1 + m_2\gamma_2}(z_-)$$

→ **Allows to express** $\Omega(\gamma; z_+)$ **in terms of** $\Omega(\gamma; z_-)$

→ **The KS-formula ensures that the complex symplectic structure on** $\mathcal{Z}'$ **is unchanged across the wall**

The Kontsevich-Soibelman wall-crossing formula

The formula:

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Simplest example: the pentagon identity

Two charges, with $\langle \gamma_1, \gamma_2 \rangle = 1 \qquad \Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = 1$

$$\longrightarrow \quad U_{\gamma_1} U_{\gamma_2} = U_{\gamma_2} U_{\gamma_1 + \gamma_2} U_{\gamma_1}$$

(e.g. SU(3) Argyres-Douglas theory)

Back to the moduli space metric

Due to the jumps in $\Omega(\gamma; z)$, the one-inst approx is discontinuous:

$$g \sim g^{\rm sf} + \Omega(\gamma; z) e^{-R|Z_\gamma(z)| + 2\pi i (q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)}$$

But multi-instanton contributions ensure that the exact metric is smooth.

The Darboux coordinates of the exact metric follow from: [Gaiotto, Moore, Neitzke]

(also [Alexandrov, Pioline, Saueressig, Vandoren])

$$\mathcal{X}'_\gamma = \mathcal{X}'^{\rm sf}_\gamma \exp\left[\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{{\rm d}\zeta'}{\zeta'} \frac{\zeta + \zeta'}{\zeta - \zeta'} \log\left(1 - \mathcal{X}'_{\gamma'}(\zeta')\right) \right]$$

The resulting metric is smooth precisely when $\Omega(\gamma; z)$ satisfies the KSWCF

Wall-crossing in Calabi-Yau string vacua

Γ is the D-brane charge lattice ($H^3(X,\mathbb{Z})$ or $H^{\text{even}}(X,\mathbb{Z})$)

$\mathcal{M}$ is a QK-manifold

$\rightarrow$ hypermultiplet moduli space of type II on X or
vector multiplet moduli space of type II on $X \times S^1$

Wall-crossing in Calabi-Yau string vacua

Γ is the D-brane charge lattice ($H^3(X,\mathbb{Z})$ or $H^{\rm even}(X,\mathbb{Z})$)

$\mathcal{M}$ is a QK-manifold

Let us fix $\mathcal{M}$ as the HM moduli space of type IIA on X for definiteness.

$\mathcal{M}$ is coordinatized by $\{X^\Lambda, \phi, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma\}$

complex structure moduli of X

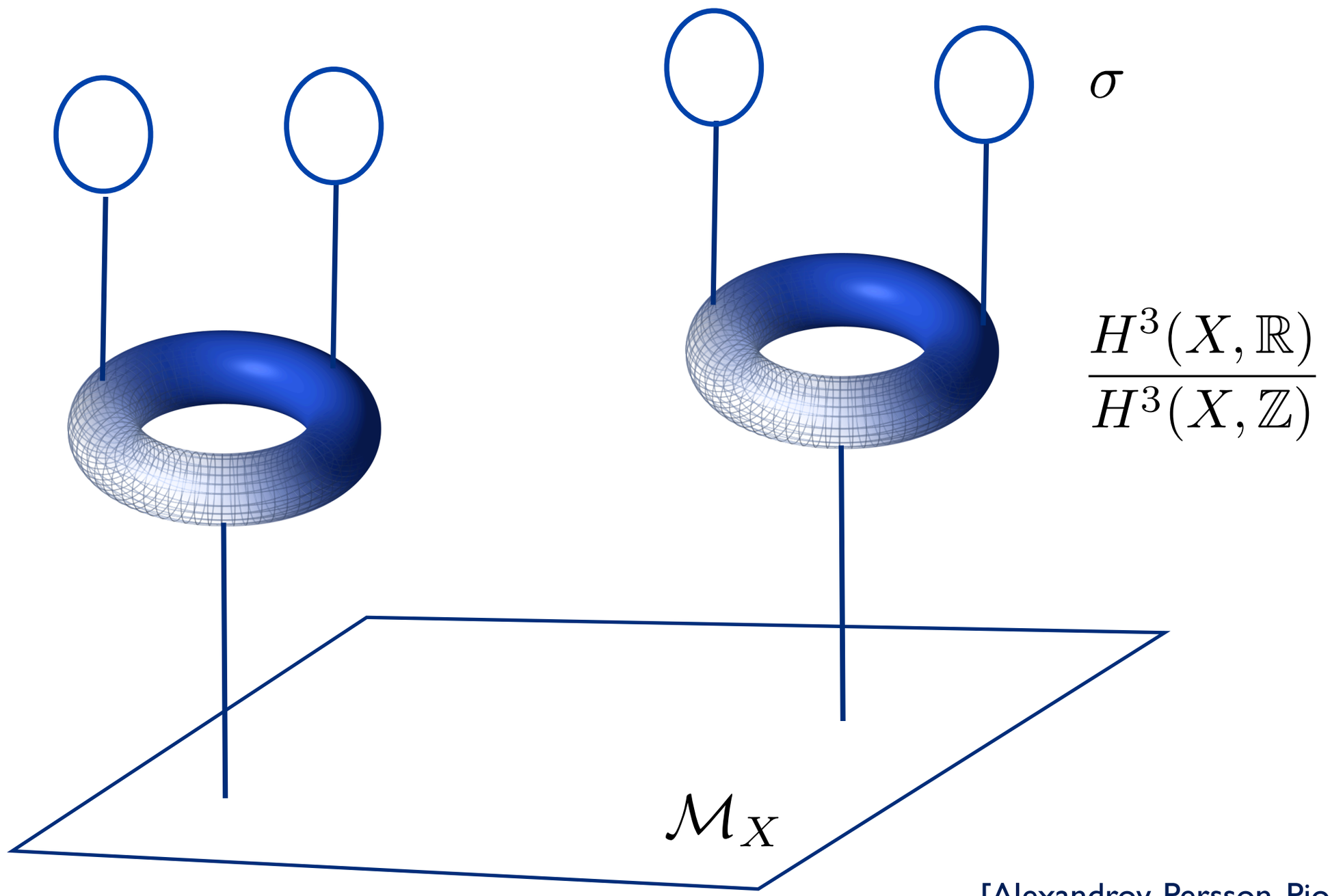
dilaton

RR-scalars

NS-axion

$$(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in H^3(X,\mathbb{R})/H^3(X,\mathbb{Z})$$

Topology of $\mathcal{M}$ at fixed small value of $g_s = e^{\phi}$ (weak-coupling) is a twisted circle bundle



[Alexandrov, Persson, Pioline]
[Baarsma,Stienstra,van der Aalst,Vandoren]

→ **Translations of $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ broken by D2-brane instantons**

→ **Translations of the NS-axion σ broken by NS5-instantons**

The analogue of the semi-flat metric is the 1-loop corrected c-map metric:

$$g \sim g^{\text{c-map}} + \Omega(\gamma) e^{-\frac{|Z_\gamma|}{g_s} + 2\pi i (q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)} + c_k e^{-\frac{|k|}{g_s^2} - i\pi k \sigma}$$

D2-instantons

NS5-instantons
(exponentially suppressed)

→ **Translations of $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ broken by D2-brane instantons**

→ **Translations of the NS-axion σ broken by NS5-instantons**

The analogue of the semi-flat metric is the 1-loop corrected c-map metric:

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In the absence of NS5-effects, $\mathcal{M}$ retains one continuous isometry ∂_σ !

The isometry ∂_σ of $\mathcal{M}$ lifts to a holomorphic action ∂_α on $\mathcal{Z}$

QK/HK correspondence valid!

A complex contact transformation V_γ of $\mathcal{Z}$ decomposes into a symplectomorphism of $\mathcal{Z}'$ and a "gauge transformation" of the section

$$\Upsilon(\zeta) = e^{\pi i \alpha(\zeta)}$$

V_γ **must also preserve the connection on** $\mathscr{L}_{\mathcal{Z}'}$

Putting all this together, one finds:

$$V_\gamma : (\mathcal{X}_{\gamma'}, \Upsilon) \longmapsto \left(U_\gamma \cdot \mathcal{X}_{\gamma'}, \Upsilon\, e^{\pi i \Delta_\gamma \alpha}\right)$$

KS-symplectormorphism

$$\Delta_\gamma \alpha = \Omega(\gamma) L(\mathcal{X}_\gamma)$$

with $L(x)$ **being the Rogers dilogarithm:**

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\,\log(1-x)$$

For a consistent construction of the line bundle $\mathscr{L}_{Z'}$ the new transformations V_γ should satisfy a "contact version" of the KS wall-crossing formula:

$$\prod_{\substack{m_1 \geq 0, m_2 \geq 0, \\ m_1/m_2 \downarrow}} V_{m_1\gamma_1 + m_2\gamma_2}(z_+) = \prod_{\substack{m_1 \geq 0, m_2 \geq 0, \\ m_1/m_2 \uparrow}} V_{m_1\gamma_1 + m_2\gamma_2}(z_-)$$

It is useful to rewrite this as:

$$\prod_s V_{\gamma_s}^{\epsilon_s} = 1$$

For the total gauge transformation on Υ to be trivial, this requires:

$$\Delta\alpha = \sum_s \epsilon_s \Omega(\gamma_s) L\big(\mathcal{X}_{\gamma_s}\big) = 0$$

(The same line bundle was recently constructed independently by A. Neitzke.)

$$\Delta\alpha = \sum_s \epsilon_s \Omega(\gamma_s) L\big(\mathcal{X}_{\gamma_s}\big) = 0$$

Very non-trivial identity for the Rogers dilogarithm!

Simplest example: the pentagon identity:

$$L(x) - L\left(\frac{x(1-y)}{1-xy}\right) - L\left(\frac{y(1-x)}{1-xy}\right) + L(y) - L(xy) = 0$$

Remarkably, the general formula can be shown to be a consequence of the motivic Kontsevich-Soibelman wall-crossing formula

[Alexandrov, DP, Pioline] (following [Fadeev, Kashaev] [Kashaev, Nakanishi])

The motivic KS wall-crossing formula

Refined BPS-index (or PSC): $$\Omega(\gamma, q; z) = \sum_{n\in\mathbb{Z}} q^{n/2}\Omega_n(\gamma; z)$$

[Dimofte, Gukov] [Gaiotto,Moore,Neitzke]

Replace the KS-transformation U_γ by:

$$\hat{U}_\gamma = \prod_{n\in\mathbb{Z}} \left(\Psi_q(-q^{n/2}\hat{\mathcal{X}}_\gamma)\right)^{-(-1)^n\Omega_n(\gamma;z)}$$

q-dilog:

$$\Psi_q(x) = \prod_{n=0}^{\infty}\left(1+q^{n+1/2}x\right)^{-1}$$

q-torus:

$$\hat{\mathcal{X}}_\gamma\hat{\mathcal{X}}_{\gamma'} = q^{\frac{\langle\gamma,\gamma'\rangle}{2}}\hat{\mathcal{X}}_{\gamma+\gamma'}$$

The motivic KS wall-crossing formula

$$\prod_{s=1}^{N} \hat{U}_{\gamma_s}^{\epsilon_s} = 1$$

Now, we wish to take the semi-classical limit $q^{1/2} \to -1$ of this formula

Following Fadeev-Kashaev, Fock-Goncharov, Kashaev-Nakanishi:

- → **Realize a product of q-dilogs as an operator in a Hilbert space**
- → **Write the vev of the operator as a "path integral"**
- → **Semi-classical limit: saddle point of the integral**
- → **At the saddle point the integrand is** $\exp\left[\sum \text{Rogers-dilogs}\right]$
- → **motivic KS-formula, the total operator is trivial:**

$$\sum \text{Rogers-dilogs} = 0$$

The motivic KS wall-crossing formula

$$\prod_{s=1}^{N} \hat{U}_{\gamma_s}^{\epsilon_s} = 1$$

Now, we wish to take the semi-classical limit $q^{1/2} \to -1$ of this formula

In detail, this procedure yields precisely:

$$\sum_{s=1}^{N} \epsilon_s \Omega(\gamma_s) L(\mathcal{X}_{\gamma_s}) = 0$$

[Alexandrov, DP, Pioline]

Generalizes various proven and conjectural formulas of Gliozzi, Tateo, Nakanishi, Inoue, Iyama, Keller, Kuniba for the Rogers dilogarithm, corresponding to cluster algebras of non-simply laced quivers.

The smoothness of the metric on $\mathcal{M}$ across a wall requires $\mathcal{X}_\gamma$ to satisfy the same integral equation as before, while the section $\Upsilon(\zeta)$ takes the form:

$$\Upsilon(\zeta) = \exp\left[i\pi\left(\sigma + \zeta^{-1}\mathcal{W} - \zeta\bar{\mathcal{W}}\right) + \frac{1}{8\pi^2}\sum_\gamma \Omega(\gamma)\int_{\ell_\gamma} \frac{\mathrm{d}\zeta'}{\zeta'}\,\frac{\zeta+\zeta'}{\zeta-\zeta'}\,L(\mathcal{X}_\gamma)\right]$$

($\mathcal{W}$ is a known function)

[Alexandrov] [Alexandrov, DP, Pioline][Neitzke]

This determines the hyperholomorphic connection on $\mathscr{L} \to \mathcal{M}'$

$$\begin{aligned}\lambda &= \bar{\partial}^{(\zeta)}\alpha + \partial^{(\zeta)}\bar{\alpha} \\ &= 2i(\partial - \bar{\partial})K_{\mathcal{B}} - \frac{1}{4}\left(\zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda - \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda\right) + \mathrm{inst}\end{aligned}$$

Conclusions and Discussion

- **Presented a general duality between HK- and QK-manifolds with isometric circle actions.**

- **The twistorial construction of the D-instanton corrected hypermultiplet metric in N=2 string vacua can be reformulated as the construction of a certain hyperholomorphic line bundle over the dual HK-manifold.**

- **Consistency of this duality requires non-trivial identities for the Rogers dilogarithm, which follow from the motivic KS wall-crossing formula.**

- **This also revealed surprising relations with cluster algebras**

Conclusions and Discussion

→ **Main missing piece of the puzzle: NS5-branes**

Hint: Our results appear to be closely related to recent unpublished work by Fock & Goncharov on the geometric quantization of cluster varieties.

Suggests the complex torus $\mathcal{M}'_\zeta = (\mathbb{C}^\times)^{2r}$ should be identified with a so called "cluster seed torus", such that $\mathcal{M}'$ is the associated cluster variety.

The complex line bundle $\mathscr{L}_{\mathcal{Z}'}$ should then correspond to a prequantum line bundle in the geometric quantization of $\mathcal{Z}'$

The main benefit of this is that FG construct a higher rank vector bundle $\mathscr{V}_\hbar$ depending on one rational "quantization parameter" $\hbar = s/r$

Conjecture: s corresponds to the NS5-brane charge k and the leading order effects of NS5-branes are captured by sections of $\mathscr{V}_\hbar$