Instantons and M5-branes

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Mathematics and Applications of Branes in String and M-theory
Issac Newton Institute, Cambridge
talk based on:

Hee-Cheol Kim (KIAS), S.K. (SNU), Eunkyung Koh (KIAS), Kimyeong Lee (KIAS), Sungjay Lee (DAMTP),

“On instantons as Kaluza-Klein modes of M5-branes,”
JHEP1112, 031, arXiv:1110.2175

related works:
D0-brane threshold bound states: [P. Yi], [Sethi-Stern], …
instanton partition functions: [Nekrasov], [Nekrasov, Okounkov], …
topological string counting of U(1) instantons: [Iqbal, Kozcaz, Shabbir]

5d SYM & 6d (2,0): [Lambert, Papageorgakis, Schmidt-Sommerfeld] [Douglas]
self-dual string anomalies: [Berman, Harvey]
DLCQ description of M5’s: [Aharony, Berkooz, Seiberg] [Lambert, Richmond]
Motivation

• Mysterious aspects of M-theory: 10d→11d, M2- and M5-branes, no perturbative description, …

• Some issues understood to certain extent: M2’s & AdS$_4$/CFT$_3$ [Bagger, Lambert] [Gustavsson] [Aharony,Bergman,Jafferis,Maldacena], …, M5’s wrapped on Riemann surfaces [Gaiotto] [Alday,Gaiotto,Tachikawa], …, …

• M5-branes? Still a lot of mysteries: N$^3$ degrees of freedom on N M5, microscopic description lacking for 6d (2,0) theory…

• Compactify M5 on a circle: D4-branes, 5d maximal SYM at low energy
Aspects of 5d SYM

• 5d SYM has nonperturbative objects which see the 6th direction.

• Instanton “solitons”: D0’s on D4’s

\[ F_{\mu\nu} = \ast_4 F_{\mu\nu} \quad \text{on} \quad R^4 \]

• Visible as QFT solitons rather than extra ingredients, unlike D0’s in (perturbative) type IIA string/SUGRA.

• 5d theory could see (at least some of) its quantum aspects.

• UV finite? [Lambert, Papageorgakis, Schmidt-Sommerfeld] [Douglas] (2010)

• We study BPS quantities: not too sensitive to UV details.

• Our analysis however will be (almost) self-contained in 5d QFT.

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Threshold bound states of instantons

- Supersymmetry & BPS bounds \((i,j=1,2,3,4, \text{SO}(5)_R \text{ spinor; } l=1,2, \ldots 5 \text{ for vector})\)

\[
\{Q^i_M, Q^j_N\} = P_\mu (\Gamma^\mu C)_{MN} \omega^{ij} + i \frac{8\pi^2 k}{g_{YM}^2} C_{MN} \omega^{ij} - i \text{tr}(qv_I) (\Gamma^I C)_{MN}
\]

\[
M \geq \frac{8\pi^2 k}{g_{YM}^2} + \text{tr}(qv_5)
\]

- Threshold bound states (with 0 binding energy) are subtle.

- Continuum: 1-particle bound & multi-particles at rest have same energy. Small relative kinetic energy makes a continuum. translation 0-modes

- Instantons also have internal noncompact 0-modes: instanton sizes

- Calculating Witten index is subtle for threshold bounds. [P.Yi] [Sethi,Stern]
The Coulomb phase index

- Coulomb phase: VEV to a scalar, $U(N) \rightarrow U(1)^N$. Instanton/W-boson bounds (dyonic instantons [Lambert,Tong]) lift to self-dual strings on M5 wrapping $S^1$ with momenta

- Size 0-modes lifted in the Coulomb phase (instantons tend to shrink)
- Translation 0-modes lifted by SO(4) chemical potential, or Omega background. (works for D0-D4, but not completely for D0 [Moore,Nekrasov,Shatashvili])
An index (continued)

• \(1/4\)-BPS: we only need 2 real SUSY to define Witten index

\[
16 \text{ SUSY } Q^i_\alpha, \bar{Q}^i_\dot{\alpha} \text{ on D4} \rightarrow 8 \text{ SUSY } \bar{Q}^i_\dot{\alpha} \text{ on D0-D4} \rightarrow 4 \text{ SUSY } \bar{Q}^a_\dot{\alpha} \text{ on D0-D4-F1} \quad (a = 1, 2)
\]

• Index:

\[
I(\mu^i, \gamma_1, \gamma_2, \gamma_3) = \text{Tr} \left[ (-1)^F e^{-\beta Q^2} e^{-\mu_i \Pi_i} e^{-i\gamma_1 (2J_{1L}) - i\gamma_2 (2J_{2L}) - i\gamma_3 (2J_{3L})} \right]
\]

- \(\text{SU(2)}_L\) of spatial SO(4)
- \(\text{SU(2)}_L\) of internal SO(4) in SO(5)
- \(\text{U(1)}^N\) electric charges
- Sum of two \(\text{SU(2)}_R\) charges in two SO(4)
- Preserves two SUSY \(\bar{Q}^+_\dot{\alpha}, \bar{Q}^-_{\dot{\alpha}}\)

• Spatial chemical potentials: “masses” to translation 0-modes

• Counts all (single/multi-particle) BPS states: careful study of dependence will let us to separate out single particle states

• \(\text{U(1)}^N\) chemical potentials: lifts internal 0-modes & size moduli.

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Path integral & index

- Path integral rep. of D0-D4 quantum mechanics (same result from QFT)
- Many ways to calculate: e.g. index invariant under following deformation:

  \[
  \text{non-commutativity (FI-term on D0-D4 mechanics, a UV regulator): } \zeta \to \infty \quad \text{regulator } \beta \to 0
  \]

- Dominated by Gaussian path integrals over a set of saddle points

- The result is actually a well-known one: Nekrasov's instanton partition function for 5d N=2\* theory on \( \mathbb{R}^4 \times S^1 \)

\[
q = e^{2\pi i \tau}: \text{instanton } \neq \text{fucacity}
\]

\[
a_i = \frac{\mu_i}{2}, \quad \epsilon_1 = i \frac{\gamma_1 - \gamma_R}{2}, \quad \epsilon_2 = i \frac{\gamma_1 + \gamma_R}{2}, \quad m = \frac{i \gamma_2}{2}
\]

- scalar VEV
- Omega background
- hypermultiplet mass

(all parameters made dimensionless by multiplying \( \beta \))

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Path integral & index (continued)

- Saddle points: solution to (generalized) ADHM, N-colored Young diagrams

\[
[\phi, B_1] = \frac{i(\gamma_R - \gamma_1)}{\beta} B_1, \quad [\phi, B_2] = \frac{i(\gamma_R + \gamma_1)}{\beta} B_2
\]

\[
[B_1, B_2] + \bar{x}^- x^+ = 0, \quad [B_1^\dagger, B_1] + [B_2^\dagger, B_2] + \bar{x}^+ x^+ - \bar{x}^- x^- = \zeta
\]

- After evaluating Gaussian path integrals, one obtains

\[
I\{y_1, y_2, \ldots, y_N\} = \prod_{i,j=1}^{N} \prod_{s \in Y_i} \frac{\sinh \frac{E_{ij} - i(\gamma_2 + \gamma_R)}{2}}{\sinh \frac{E_{ij}}{2}} \frac{\sinh \frac{E_{ij} + i(\gamma_2 - \gamma_R)}{2}}{\sinh \frac{E_{ij} - 2i\gamma_R}{2}}
\]

\[
E_{ij} = \mu_i - \mu_j + i(\gamma_1 - \gamma_R) h_i(s) + i(\gamma_1 + \gamma_R) (v_j(s) + 1)
\]

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Pure instanton bound states

- M-theory demands $k$ U(1) instantons form unique threshold bound state (1 supermultiplet): KK modes for a free 6d tensor multiplet

- Direct study: normalizable harmonic forms on instanton moduli space, very difficult for higher $k$, only studied up to $k=2$ [K.Lee, D.Tong, S.Yi]

Index (rewrite using topological vertex technique) [Iqbal, Kozcaz, Shabbir]

$$I = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} I_{sp}(q^n, n\gamma) \right], \quad I_{sp} = \left( \frac{\sin \frac{\gamma_1+\gamma_2}{2} \sin \frac{\gamma_1-\gamma_2}{2}}{\sin \frac{\gamma_1+\gamma_R}{2} \sin \frac{\gamma_1-\gamma_R}{2}} \right) \frac{q}{1-q} \equiv I_{com}(\gamma_1, \gamma_2, \gamma_R) \frac{q}{1-q}$$

index from the c.o.m. 0-modes & fermion 0-modes

- One c.o.m. supermultiplet at $O(q^k)$: 5d QFT derivation of uniqueness
Charged instanton bound states: SU(2)

- Chemical potential $e^{-(\mu_1 - \mu_2)}$ measures # of W-bosons or self-dual strings.

- Single particle index for one self-dual string:

$$z_{sp} = 1 + 8q + 40q^2 + 160q^3 + 552q^4 + 1712q^5 + 4896q^6 + 13120q^7 + 33320q^8 + 80872q^9 + 188784q^{10} + \cdots$$

  set $\gamma_2 = \pi$: kills $(-1)^F$ sign

  set $\gamma_R = 0$, $\gamma_1 \to 0$ and factor out $I_{com}$

- This can be written in a closed form as:

$$z_{sp} = I_{com} \prod_{n=1}^{\infty} \frac{(1 + q^n)^4}{(1 - q^n)^4}$$

  or...

$$\left(\frac{\sin \gamma_2 + \gamma_R}{2} \cdot \frac{\sin \gamma_2 - \gamma_R}{2}\right) \prod_{n=1}^{\infty} \frac{(1 - q^n e^{i(\gamma_2 + \gamma_R)})(1 - q^n e^{i(\gamma_2 - \gamma_R)})(1 - q^n e^{i(-\gamma_2 + \gamma_R)})(1 - q^n e^{i(-\gamma_2 - \gamma_R)})}{(1 - q^n e^{i(\gamma_1 + \gamma_R)})(1 - q^n e^{i(\gamma_1 - \gamma_R)})(1 - q^n e^{i(-\gamma_1 + \gamma_R)})(1 - q^n e^{i(-\gamma_1 - \gamma_R)})}$$

- 1+1d fluctuations on 0-modes of an SU(2) self-dual string: instantons again form a KK tower of M5, all visible from 5d QFT framework.

- Our index also predicts nontrivial bounds of multiple self-dual strings
Charged bound states: SU(N)

- Physics to expect is less clear:
  - Single W-bosons ending on i & j’th D4’s: SU(j – i + 1) matters.

- Partition function for the “longest string” in SU(N)

\[
I_{sp}^{SU(3)} = I_{com} \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 - q^n} \right)^4 (1 + 16q + 96q^2 + 448q^3 + 1728q^4 + 5856q^5 + \cdots)
\]

\[
I_{sp}^{SU(4)} = I_{com} \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 - q^n} \right)^4 (1 + 32q + 448q^2 + 3968q^3 + 27008q^4 + \cdots)
\]

\[
I_{sp}^{SU(5)} = I_{com} \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 - q^n} \right)^4 (1 + 48q + 1056q^2 + 14656q^3 + 149568q^4 + \cdots)
\]

- For SU(N), general closed form for the ‘relative part’ of the index is

\[
\int \frac{dz}{2\pi i z} \prod_{n=1}^{\infty} \left( \frac{1 + q^{2n-1}z}{1 - q^{2n-1}z} \right) \left( \frac{1 + q^{2n-1}z^{-1}}{1 - q^{2n-1}z^{-1}} \right)^2 \left[ \frac{2q^{1/2}z}{1 - q} \left( z + \frac{1}{z} \right) \right]^{N-2}
\]

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Charged SU(N) bound states (continued)

• Possible interpretation: degrees of freedom living on the worldvolume of self-dual strings $\sim (j - i - 1)$ of them for ij string.

• Not all “physical”: $\text{U}(1)^{j-i-1}$ singlet conditions

• For large momentum ($\sim 6d$ (2,0) limit, or self-dual string on a large circle), the last constraint effectively becomes irrelevant (saddle point at $z=1$)

• 1+1d theories for $N^2$ self-dual strings have more degrees than just giving momenta to the $N^2$ 0-modes: So how many 1+1d “degrees” do we find?
Degrees of freedom on self-dual strings?

• M5-branes in symmetric phase should exhibit $N^3$ degrees of freedom.

• Coulomb phase sometimes shows remnants of the symmetric phase degrees as massive states: e.g. $N^2$ massive BPS W-bosons of SYM

• Self-dual strings on $S^1$: $N^2$ with 0 momentum. Large momentum regime?

• Previous 2d d.o.f. are effectively “unconstrained” in UV

• Count all of them, giving $\frac{1}{2}$ contribution to fermions

$$\#_{B}^{\text{int}} = \#_{F}^{\text{int}} = 4N C_3 = \frac{2}{3} N(N-1)(N-2)$$

$$\#_{B}^{\text{ext}} = \#_{F}^{\text{ext}} = 4N C_2 = 2N(N-1)$$

$$\#_{B} + \frac{1}{2} \#_{F} = N(N^2 - 1)$$
The instanton index in the symmetric phase

- U(N) unbroken if VEV=0. Still, one can turn on U(1)^N chemical potentials. Regularize divergent volume of internal moduli. (similar to SO(4) Ω-background)

- Technically, same path integral & same expression: so what is this…?

- Non-relativistic superconformal sigma model in IR: DLCQ M5’s [Aharony, Berkooz, Kachru, Seiberg, Silverstein] [Aharony, Berkooz, Seiberg] [Lambert, Richmond]

- 6d superconformal generators commuting with \( P_- = M_{6-} + M_{7-} \)

\[
x^\pm = x^0 \pm x^5
\]

SO(6,2) :

\[
M_{AB}, \ A, B = 0, 1, 2, \cdots, 6, 7
\]

\[
P_\mu = M_{6\mu} + M_{7\mu} , \quad K_\mu = -M_{6\mu} + M_{7\mu} , \quad \Delta = M_{67}
\]

\[\rightarrow H \sim P_+ , \quad P_i , \quad M_{ij} , \quad G_i \sim M_{-i} , \quad K \sim K_- , \quad D = \Delta - M_{05}\]

can be extended to Osp(6,2|4): 32 SUSY reduced to 24

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The non-relativistic superconformal index

- One can count BPS local operators in NR quantum mechanics.
- Equivalently, one can count BPS states (non-relativistic operator-state map):

  \[ M^{-1}(iD)M = H + K = L_0 \]
  \[ M = e^{H/2}e^{-K} \]

  \[ M^{-1}QM = Q - iS \equiv \hat{Q}, \quad M^{-1}SM = -i/2(Q + iS) = -\frac{i}{2}\hat{S} \]

  \[ \{\hat{Q},\hat{S}\} = L_0 + (4J_{2R} + 2J_{1R}) \]

- Our index is a **nonrelativistic superconformal index** [Nakayama]

\[
I_{SC} = \text{Tr} \left[ (-1)^F e^{-\beta\{\hat{Q},\hat{S}\}} e^{-2i\gamma_R J_R} e^{-2i\gamma_1 J_{1L}} e^{-2i\gamma_2 J_{2L}} e^{-i\alpha_i \Pi_i} \right]
\]

- \(SU(2)_R\) charge \(\sim\) BPS dilatation eigenvalues
- \(U(N)\) chemical potentials: project to \(U(N)\) singlets (gauge invariant operators)

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Some tests

• Tests: U(1) instantons vs. DLCQ index of single M5 (rather trivial)

• Large N index at k=1: SUGRA index on AdS$_7 \times S^4$ (DLCQ basis in AdS)

\[ I_{k=1} = \frac{e^{i\gamma_2} + e^{-i\gamma_2} - e^{i\gamma_1} - e^{-i\gamma_1}}{(1 - te^{i\gamma_1})(1 - te^{-i\gamma_1})} \left[ \sum_{n=0}^{N-1} \chi_{n+1/2}(\gamma_2)t^{n+1} - \sum_{n=1}^{N-1} \chi_{n-1/2}(\gamma_2)t^{n+2} \right] \quad t \equiv e^{-i\gamma_R} \]

\[ I_{N \to \infty, k=1} = \frac{e^{i\gamma_2} + e^{-i\gamma_2} - e^{i\gamma_1} - e^{-i\gamma_1}}{(1 - te^{i\gamma_1})(1 - te^{-i\gamma_1})} \frac{t - t^3}{(1 - te^{i\gamma_2})(1 - te^{-i\gamma_2})} = I_{\text{SUGRA}, k=1} \]

| $p \geq 1$ | $2p$ | 0 | $\frac{p}{2}$ | $p$ | \begin{tabular}{c} boson/fermion \\ b \end{tabular} |
| $p \geq 1$ | $2p + 1$ | 0 | $\frac{p-1}{2}$ | $p + 1$ | f |
| $p \geq 1$ | $2p$ | $\frac{1}{2}$ | $\frac{p-1}{2}$ | $p$ | f |
| $p \geq 2$ | $2p + 1$ | $\frac{1}{2}$ | $\frac{p-2}{2}$ | $p + 1$ | b |
| $p \geq 2$ | $2p$ | 0 | $\frac{p-2}{2}$ | $p$ | b |
| $p \geq 3$ | $2p + 1$ | 0 | $\frac{p-3}{2}$ | $p + 1$ | f |
| $\vdots$ | 3 | 0 | 0 | 2 | b (fermionic constraint) |

• Similar agreement at k=2 (& presumably true with higher momentum)
Concluding remarks

- Instantons reveal interesting information on $(2,0)$ theory on circle via index: in both Coulomb & symmetric phases

- Microscopic studies of conformal 6d $(2,0)$: 6d limit (large instanton number $k$) & beyond low E. For instance, $N^3$ is beyond SUGRA & maybe at large $k$.

- Self-dual string anomaly ($SO(4)_N$ in $SO(5)$ from Coulomb branch) [Berman, Harvey]

For $G = SU(N)$, $H = U(1)^{N-2}$,

$$c = \frac{\alpha_e}{2} = \frac{N^2 - N}{8} \text{ from } N - 1 \text{ different strings}$$