Definable relations in the
Turing degree structures

M.M. Arslanov
Department of Mathematics,
Kazan Federal University, Kazan, RUSSIA

Marat.Arslanov@ksu.ru
The finite level \( n, n \geq 1 \), of the Ershov hierarchy constitutes \( n \)-c. e. sets which can be presented in the canonical form as

\[
A = \left[ \frac{n-1}{2} \right] \bigcup_{i=0} \{ (R_{2i+1} - R_{2i}) \cup (R_{2i} - R_{2i+1}) \}
\]

for some c. e. sets \( R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_{n-1} \).
(Here if \( n \) is an odd number then \( R_n = \emptyset \).)

\( R_0, \; R_1 - R_0, \; (R_2 - R_1) \cup R_0, \ldots \)
A Turing degree $a$ is $n$-c. e. if it contains an $n$-c. e. set, and it is a properly $n$-c. e. degree if it contains an $n$-c. e. set but no $(n - 1)$-c. e. sets.

We denote by $\mathcal{D}_n$ the partial ordered set of all $n$-c. e. degrees and by $\mathcal{D}$ the class of all Turing degrees.

$\mathcal{R} = \mathcal{D}_1$ denotes the set of c. e. degrees.

$\mathcal{R} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots \subset \mathcal{D}_n \subset \ldots \subset \mathcal{D}(\leq 0')$
Theorem (Erschov Ju. L.)

\[ \{ A \mid A \leq_T \emptyset' \} = \bigcup_{a \in \mathcal{O}} \Sigma^{-1}_a = \]
\[ = \bigcup_{a \in \mathcal{O}, |a|_0 = \omega^2} \Sigma^{-1}_a \cdot \]

\[ \text{all degrees } \leq 0' \]

\[ \text{c. e. degrees} \]

0'
Open question

- Definability of the various levels of the n-c.e. degree hierarchy, both relatively and within wider local structures; more specifically, questions related to the definability of the relations of ‘computably enumerable’ and ‘computably enumerable in’;
Let $\mathcal{E}_n, 1 \leq n < \omega$, be the class of all $n$-c.e. sets. $\tilde{\mathcal{E}}_n, 1 \leq n < \omega$, denotes the class of all co-$n$-c.e. sets.

Let $\mathcal{E}_\infty = \{\bigcup_{1 \leq n < \omega} \mathcal{E}_n; \bigcup, \bigcap, \omega, \emptyset\}$.

$\mathcal{E} = \mathcal{E}_1$ denotes the set of all c.e. sets.

We denote $\mathcal{E}^* = \mathcal{E}/\approx^*$ (similarly for all others),

where $A \equiv^* B$ iff $(A - B \cup (B - A))$ finite.

Let $\{V_{n,e}\}_{e \in \omega}$ denotes an effective enumeration of all $n$-c.e. sets, $n \geq 2$. 
We say that a set of Turing degrees $\mathcal{C}$ is definable in $\mathcal{E}_\infty$ (in $\mathcal{E}_n$ for some $n \geq 1$) if there is a definable in $\mathcal{E}_\infty$ (in $\mathcal{E}_n$) class of sets $S \subset \mathcal{E}_\infty$ such that
\[
\mathcal{C} = \{\deg B \mid B \in S\}.
\]

*Example.* The degree 0 definable in each $\mathcal{E}_\alpha$, $1 \leq \alpha \leq \omega$: $0 = \deg(\emptyset)$. 
Let $\mathcal{D}$, $\mathcal{D}(\leq 0')$ and $\mathcal{D}_n, 1 \leq n < \omega$, be the partial ordered structures of all Turing degrees, all Turing degrees $\leq 0'$, and all $n$-c. e. degrees, accordingly.

$\mathcal{R} = \mathcal{D}_1$ denotes the set of c. e. degrees.
We say that a set of Turing degrees $\mathcal{C}$ is definable in a structure $\mathcal{D}_n$, if there is a formula $\varphi(x)$ in its language $\mathcal{L} = \{\leq\}$ such that

$$\mathcal{D}_n \models \varphi(a) \iff a \in \mathcal{C}$$
Part 1. Definability of classes of degrees in $\mathcal{E}_n$, $\mathcal{E}_\infty = \{ \bigcup_{1 \leq n < \omega} \mathcal{E}_n; \cup, \cap, \omega, \emptyset \}$.

Let for each $n > 0$,

$$H_n = \{ \text{a c.e.} \mid a^n = 0^{(n+1)} \},$$

$$L_n = \{ \text{a c.e.} \mid a^n = 0^n \}.$$
**Theorem.** In $\mathcal{E} = \mathcal{E}_1$

a) all classes $H_m, m \geq 1$, are definable (Martin ($m = 1$), Cholak-Harrington ($m > 1$)). and

b) all classes $\overline{H}_m, m \geq 1$, are non definable (Harrington-Soare, Cholak-Downey-Stob).
**Theorem.** In $\mathcal{E}$

a) all classes $L_n, n \geq 1$, are not definable (Harrington-Soare, Cholak-Downey-Stob).

and

b) all classes $\bar{L}_n, n > 1$, are definable (Lachlan,Shoenfield ($n = 2$), Cholak-Harrington ($n \geq 2$).

c) $\bar{L}_1$ not definable (R.Epstein).
In case of n-c.e. sets ($n > 1$):

**Theorem** The class of all computable sets definable in each $\mathcal{E}_\alpha, 1 \leq \alpha \leq \omega$ by formula $\varphi(A) :=$

\[ A = \omega \lor \exists Z (\bar{A} \cap Z \neq \emptyset \land \bar{A} \cap \bar{Z} \neq \emptyset) \land (\forall X \exists Y_0, Y_1 (X = Y_0 \cup Y_1 \land Y_0 = X \cap A)) \]
**Theorem.** For each $n > 0$, the class of sets $\delta_n := \mathcal{E}_{n+1} \cap \mathcal{E}_{n+1}$ definable in $\mathcal{E}_{n+1}$.
**Theorem** (Lempp, Nies) The class of c.e. sets $\mathcal{E}$ definable in $\mathcal{E}_2$.

*Corollary.* The set of c.e. degrees definable in $\mathcal{E}_2$. 
**Theorem.** The set of all high c.e. degrees $H_1$ definable in each $\mathcal{E}_n$, 
$1 \leq n < \omega$. 
Proof.

• n-c.e. cohesive sets are definable in $\mathcal{E}_n$ for each $n > 1$.

• Any n-c.e. cohesive set is co-c.e.

• $H_1 = \{\deg(V_{2n,e}) : V_{2n,e} \text{ cohesive}\}$. 
**Theorem.** For each $n > 1$ there exists a high properly $n$-c.e. degree.

**Theorem.** For each $n > 1$ there exists a definable in $\mathcal{E}_{n+1}$ set of $n$-c.e. degrees $C$ such that $C$ contains all high $n$-c.e. degrees.
Theorem. For any given $n \geq 1$, a) the class $\mathcal{E}_n$ of all $n$-c. e. sets is not computably presentable, i. e. it is not isomorphic to any computable partial ordering.
Part 2. Definability in the degrees.

Now let

\[ \mathcal{D}_n = \{ \text{n-c. e. degrees}; \leq_T \}. \]

If \( n = 1 \) (the c.e. degrees):

- All classes \( H_n \) and \( L_n \) (except possibly \( L_1 \)) are definable.)

(Nies, Shore and Slaman)
The formula
\[ \varphi(x) = (\exists y > x)(\forall z \leq y)(z \leq x \lor x \leq z) \]
defines in \( D_2 \) an infinite set of c.e. degrees.

(A., Kalimullin, Lempp)
Theorem. a) (Yamaleev) For any properly d-c.e. degree and for any nontrivial splitting of $d = d_0 \cup d_1$ into d-c.e. degrees $d_0$ and $d_1$, for an $i \leq 1$ any d-c.e. degree $u_i$, $d_i < u_i \leq d$, is splittable in d-c.e. degrees avoiding the upper cone of $d_i$.

b) There is a c.e. degree $c > 0$ and a nontrivial splitting of $c = c_0 \cup c_1$ into d-c.e. degrees $c_0$ and $c_1$ such that for each $i \leq 1$ there is a d-c.e. degree $d_i, c_i < d_i < c$, which is not splittable into d-c.e. degrees avoiding the upper cone of $c_i$. 
Let $C$ denotes the (definable) class of c.e. degrees which have the property $b$) of this theorem.

**Low Density Conjecture:** The class $C$ dense in the low c.e. degrees:

$(\forall \text{ low c.e. } a < b)(\exists c \in C)(a < c < b)$?
**Theorem (Sacks).** Every noncomputable c.e. degree \( a \) is nontrivially splittable into c.e. low degrees:
\[
a = a_0 \cup a_1, \quad a'_0 = a'_1 = 0', \quad \text{and} \quad a_0 \mid a_1.
\]

Low density \( \vdash \) Low splitting \( \rightarrow \) definability of c.e. degrees in \( \mathcal{D}_2 \):
For any given c.e. degree $a > 0$

- First split $a$ into two low c.e. degrees $a_0$ and $b_0$: $a = a_0 \cup b_0$
- Then find low c.e. degrees $a_1 > a_0$ and $b_1 > b_0$,
- Then using low density theorem find $C$-degrees $c_0$ and $c_1$ such that $a_0 < c_0 < a_1$ and $b_0 < c_1 < b_1$.

We have $a = c_0 \cup c_1$. 
Theorem. Let $a > 0$ be a high c.e. degree ($a' = 0''$). Then there exists a c.e. degree $c \leq a$ and a nontrivial splitting of $c$ into d-c.e. degrees $c_0$ and $c_1$ such that for each $i \leq 1$ there exists a d-c.e. degree $d_i, c_i < d_i < c$, which is not splittable into d-c.e. degrees avoiding the upper cone of $c_i$. 
Open question: Can we change in this theorem the condition $a' = 0''$ (a is a high degree) by the condition $a' = 0'$ (a is a low degree)?
Theorem. Let $a > 0$ be a low $n$-c.e. degree for some $n \geq 1$. Then the set of $n$-c.e. degrees $\{b \mid b \leq a\}$ is definable from parameters in $\mathcal{D}(\leq 0')$. 
It follows from the following technical

**Theorem.** For any low n-c.e. degree \(a > 0\) and for any \(n, 1 \leq n < \omega\), the set of all n-c.e. degrees \( \{b \mid b \leq a\} \) forms a uniformly low set.

(A set of degrees \(\mathcal{A}\) is uniformly low if there exists a sequence of sets \(\langle X(n) \mid n \in \omega \rangle\) which are representatives for the degrees in \(\mathcal{A}\) (i.e. \(\{\text{deg}(X(n)) \mid n \in \omega\} = \mathcal{A}\)) and there is a \(\emptyset'\)-computable function \(f\) such that \(\Phi_{f(n)}^{\emptyset'} = (X(n))'\).)
• (Slaman, Woodin). Every uniformly low set of $\Delta^0_2$-degrees, bounded by a low degree $a$, is definable from parameters in $\mathcal{D}(\leq 0')$. 
Open questions:

• Which other natural sets of $n$-c.e. degrees definable in the $\Delta^0_2$-degrees.

• Is the set of all $n$-c.e. degrees for some $n > 1$ definable from parameters in the $\Delta^0_2$-degrees?

• Is the set of all c.e. degrees definable from parameters in the $n$-c.e. degrees for some /each $n > 1$?
**Theorem** Classes of all c. e. and 2-c.e. sets are definable in \( \{\mathcal{D}_n, \leq, \text{CEIN}\} \) for each \( n \geq 3 \).

(Here \( \text{CEIN}(x,y) = "x \text{ is c. e. in } y" \))

**Proof.**

a) \( x \) is c. e. iff \( x = \text{CEIN}(0) \),
b) \( x \) is 2-c.e. iff

\( (\exists y)(y \text{ is c. e., } y \leq x \text{ and } x = \text{CEIN}(y)) \).

(By Arslanov, LaForte and Slaman, if a n-c.e. \( (n \geq 3) \) degree a \( \text{CEIN}(b) \) for a c.e. degree \( b \), then \( a \) is a d-c.e. degree.)
Open question. Is this true for the classes of n-c.e. degrees for $n > 2$?
Theorem for any $n \geq 1$,

a) for any $n$-c.e. degree $a > 0$ $D_n(\leq a)$ is not computably presentable.

b) $D_n$ is not countably categorical.